The Ito Formula for Essentially Self-adjoint Quantum Semimartingales

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1 Introduction.

It is a source of embarrassment that the quantum Ito formula

$$f(\hat{M}_t) \quad = \quad f(\hat{M}_0) + \int_0^t (Df(\hat{M}_s)(d\hat{M}_s) + D_I^2 f(\hat{M}_s)(d\hat{M}_s, d\hat{M}_s)) \tag{1}$$

in [28], for a regular (bounded) quantum semimartingale M, does not directly imply the Ito formula for classical Brownian motion W:

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\, dW_s + \frac{1}{2}\int_0^t f''(W_s)\, d\left\langle W\right\rangle_s \tag{2}$$

This is unsatisfactory from our present point of view, which is to derive as much as possible of the classical theory of Brownian motion from the theory of quantum stochastic processes. The quantum Ito formula belongs to non-commutative probability. It is a purely operator theoretic result and makes no reference to paths. It implies classical results by regarding random variables as multiplication operators and using the Wiener–Ito isomorphism between Weiner space and Fock space.

In this article we find a remedy and show that (2), and its extension to gauge processes, includes the case where $\hat{M}$ is an essentially self-adjoint quantum semimartingale.

The classical Ito formula (2) then follows from the quantum Ito formula (1) as in [28, Section 7.] by taking $\hat{M} = B \equiv A + A^\dagger$, the Brownian quantum semimartingale, which is a process of unbounded essentially self-adjoint operators.

In (1) the process $f(\hat{M}) = \{\hat{M}_t : t \in [0,1]\}$ is defined by the Fourier functional calculus, $Df(X)(\cdot)$ is the differential of f and $D_I^2 f(X)(\cdot,\cdot)$ is the (unsymmetrised) 'Ito' second differential of f at the operator X.

The Ito formula (2) may be obtained indirectly from (1), quite simply, as follows. It was shown in [28] that (1) implies (2) when W is replaced by $h_n(W)$ where h_n is bounded twice continuously differentiable. The Ito formula for Brownian motion then follows by approximating the function $h(x) = x$ by an appropriate sequence of h_n, for example the sequence used in Section 13. Lifting such an argument to non-commuting essentially self-adjoint quantum semimartingales presents difficulties.

Let $\hat{M} = \{\hat{M}_t : t \in [0,1]\}$ be a symmetric quantum semimartingale:

$$\hat{M}_t = \hat{M}_0 + \int_0^t (\hat{E}\, d\Lambda + \hat{F}\, dA + \hat{F}^*\, dA^\dagger + \hat{H}\, ds). \tag{3}$$

The integrands are adapted processes of operators in $\mathcal{B}(\mathfrak{H})$, where $\mathfrak{H}$ is Bose–Fock space over $L^2[0,1]$, and E_t and H_t are self-adjoint for all t.

$\hat{M}$ is *essentially self-adjoint* with *core* $\mathcal{D}$ if $\hat{M}_t$ is essentially self-adjoint on $\mathcal{D}$ for almost all t. In this case the *closure* of $\hat{M}$ is $\overline{M} = \{\overline{M}_t : t \in [0,1]\}$ where $\overline{M}_t$ is the closure of $\hat{M}_t$. The process of bounded operators

$$e^{i\overline{M}} = \{e^{i\overline{M}_t} : t \in [0,1]\}$$

may then be defined by the functional calculus.

An essentially self-adjoint quantum semimartingale $\hat{M}$ *satisfies the quantum Duhamel formula* if $e^{i\overline{M}}$ is a regular quantum semimartingale and

$$e^{i\overline{M}_t} = I + \int_0^t (\hat{E}_{\exp(iM)}\, d\Lambda + \hat{F}_{\exp(iM)}\, dA + \hat{G}_{\exp(iM)}\, dA^\dagger + \hat{H}_{\exp(iM)}\, ds), \tag{4}$$

where

$$\begin{aligned}
\hat{E}_{\exp(iM)}(t) &= e^{i(\overline{M}_t + \hat{E}_t)} - e^{i\overline{M}_t} \\
\hat{F}_{\exp(iM)}(t) &= i\int_0^1 e^{i(1-u)\overline{M}_t} \hat{F}_t e^{iu(\overline{M}_t + \hat{E}_t)}\, du \\
\hat{G}_{\exp(iM)}(t) &= i\int_0^1 e^{i(1-u)(\overline{M}_t + \hat{E}_t)} \hat{F}_t^* e^{iu\overline{M}_t}\, du
\end{aligned} \tag{5}$$

$$\begin{aligned}\hat{H}_{\exp(iM)}(t) &= i\int_0^1 e^{i(1-u)\overline{M}_t}\hat{H}_t e^{iu\overline{M}_t}\,du \\ &\qquad + i^2\int_0^1\int_0^1 u e^{i(1-u)\overline{M}_t}F_t e^{iu(1-v)(\overline{M}_t+\hat{E}_t)}\hat{F}_t^* e^{iuv\overline{M}_t}\,du\,dv\end{aligned}$$

Let $f \in C^{2+}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : p \mapsto p^2\hat{f}(p) \text{ is in } L^1(\mathbb{R})\}$ where $\hat{f}$ is the Fourier transform of f. If $p\hat{M}$ satisfies the quantum Duhamel formula for each real p then $f(\overline{M}) = \{f(\overline{M}_t) : t \in [0,1]\}$ is a regular quantum semimartingale and *satisfies the quantum Ito formula*

$$f(\overline{M}_t) = f(0) + \int_0^t (\hat{E}_{f(M)}\,d\Lambda + \hat{F}_{f(M)}\,dA + \hat{G}_{f(M)}\,dA^\dagger + \hat{H}_{f(M)}\,ds) \qquad (6)$$

where, for $\hat{X}$ in $\{\hat{E}, \hat{F}, \hat{G}, \hat{H}\}$,

$$\hat{X}_{f(M)} = \int_{-\infty}^{\infty} \hat{f}(p)\hat{X}_{\exp(ipM)}\,dp$$

This is the case whenever M is regular and self-adjoint [28].

The original formula of Hudson and Parthasarathy will be referred to as the *quantum Ito product formula.*

It was shown in [28] that (6) implies the Ito formula for a large class of classical semimartingales, which may even have jumps. Unfortunately this class does not include Brownian motion.

The main objectives in this article are to

(a) identify a non-trivial class, $\mathcal{C}$, containing B, of essentially self-adjoint quantum semimartingales;

(b) show that if $\hat{M}$ belongs to $\mathcal{C}$ then $p\hat{M}$ satisfies the quantum Duhamel formula for all real p and $f(\hat{M})$ satisfies the quantum Ito formula for all $f \in C^{2+}(\mathbb{R})$;

(c) enlarge the class $\mathcal{C}$ identified in (a) and (b) by showing that if $\hat{M}$ is an essentially self-adjoint quantum semimartingale satisfying the quantum Duhamel formula then so is the perturbation $\hat{M} + \hat{J}$ whenever $\hat{J}$ is a regular self-adjoint quantum semimartingale.

We also touch briefly on the quantum Stratonovich formula.

We approach (a) and (b) *via* "chaos matrices".

Bose–Fock space may be defined as the direct sum of Hilbert spaces, sometimes called chaos spaces:

$$\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}^1 \oplus \cdots \oplus \mathfrak{H}^j \oplus \cdots, .$$

$\mathfrak{H}^j$ is a closed subspace of the j-fold tensor product $L^2[0,1]^{\otimes j}$. The symmetric tensors $\{f \otimes \cdots \otimes f : f \in L^2[0,1]\}$ are in $\mathfrak{H}^j$. The *exponential vectors*, $\{e(f) : f \in L^2[0,1]\}$ where

$$e(f) = 1 \oplus f \oplus \frac{f^{\otimes 2}}{(2!)^{1/2}} + \cdots + \frac{f^{\otimes j}}{(j!)^{1/2}} + \cdots,$$

are total in $\mathfrak{H}$.

A matrix T with entries $T^i_j \in \mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i)$ is called a chaos matrix. Every bounded operator in $\mathfrak{H}$ has a unique chaos matrix representation. The same is not true of unbounded operators.

The chaos matrix approach is suited to Lindsay's construction of quantum stochastic integrals *via* the gradient process ∇ and the Hitsuda–Skorohod process $\mathcal{S}$ [14]. These are processes of unbounded operators with simple superdiagonal and subdiagonal chaos matrix representations. Lindsay's integrals are defined for non-adapted processes and are an extension of Hudson and Parthasarathy's integrals.

The basic processes $\Lambda, A, A^\dagger, t$ of quantum stochastic calculus have uncomplicated chaos matrix representations. Representations leading naturally to a definition of the chaos matrix quantum stochastic integral of a chaos matrix-valued process with respect to $d\Lambda$, dA, $dA^\dagger$ and ds.

The correspondence between operator-valued and matrix processes is preserved under the respective quantum stochastic integrations. If the operators $\hat{E}_t$, $\hat{F}_t$, $\hat{F}^*_t$ and $\hat{H}_t$ in the quantum semimartingale (3) are represented by chaos matrices, E_t, F_t, F^*_t and H_t and

$$M_t = M_0 + \int_0^t (E\, d\Lambda + F\, dA + F^*\, dA^\dagger + H\, ds),$$

then M_t is the chaos matrix representation of $\hat{M}_t$ for $t \in [0,1]$.

The entries of the chaos matrix processes E, F, F^* and H are themselves operator valued processes that belong to Lebesgue spaces:

$$E^i_j \in L^\infty([0,1], \mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i)_{\mathrm{so}}), \quad F^i_j, (F^*)^i_j \in L^2([0,1], \mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i)_{\mathrm{so}}),$$
$$H^i_j \in L^1([0,1], \mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i)_{\mathrm{so}}).$$

These spaces have Bochner–Lebesgue norms but measurability is with respect to the strong operator topology on $\mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i)$.

The norms are used to define a control matrix, $\mathcal{K} = \mathcal{K}(M)$:

$$\mathcal{K}^i_j = i^{1/2} \left\| E^{i-1}_{j-1} \right\|_\infty j^{1/2} + \left\| F^i_{j-1} \right\|_2 j^{1/2} + i^{1/2} \left\| (F^*)^{i-1}_j \right\|_2 + \left\| H^i_j \right\|_1, \quad i, j = 0, 1, \ldots.$$

The control matrix plays a rôle similar to that of the dominating function in Lebesgue's dominated convergence theorem.

Let ℓ^2 be the Hilbert space of complex sequences $x = (x_0, x_1, \ldots, x_n, \ldots)^t$, written as infinite column vectors, with $\|x\|_2 = \left(\sum_{n=0}^\infty |x_n|^2\right)^{1/2} < \infty$. Denote by ℓ_{00} the dense subspace of sequences with finitely many non-zero entries.

The matrix $\mathcal{K}$ represents an operator in ℓ^2 (possibly with domain $\{0\}$.) A vector $x \in \ell^2$ is analytic for $\mathcal{K}$ if $x \in \mathcal{D}(\mathcal{K}^n)$ for $n = 1, 2, \ldots$ and

$$\sum_{n=0}^\infty z^n \frac{\|\mathcal{K}^n x\|}{n!}$$

has non-zero radius of convergence $r(x)$. The vector space of analytic vectors x with $r(x) \geq \epsilon$ is denoted $\mathcal{A}_\epsilon(\hat{\mathcal{K}})$.

From the results of Sections 10 and 11 we obtain the following theorem which allows us to achieve the objectives (a) and (b) above. Let $\mathcal{E} \subset \mathfrak{H}$ be the linear span of the exponential vectors.

Theorem 1.1 *Let $\hat{M}$ be a symmetric quantum semimartingale whose chaos matrix representation M has control matrix $\mathcal{K}$ and let $\epsilon > 0$.*

If $\ell_{00} \subset \mathcal{A}_\epsilon(\mathcal{K})$ then

(i) *$\hat{M}$ is essentially self-adjoint with core $\mathcal{E}$.*

(ii) *pM satisfies the quantum Duhamel formula for each real p.*

(iii) *$f(\overline{M})$ satisfies the quantum Ito formula for each $f \in C^{2+}(\mathbb{R})$.*

The conditions of this theorem are satisfied in particular when E, F, F^* and H are adapted processes of $(2k+1)$-diagonal chaos matrices. It is shown in Section 5 that there is a large class of such adapted processes defined by kernels.

The proof of essential self-adjointness uses Nelson's analytic vector theorem. A chaos matrix version of the quantum Ito product formula is then proved for powers of M. A chaos matrix version of the quantum Duhamel formula and the quantum Ito formula are then obtained in much the same way as for regular quantum semimartingales. The operator versions of the formulae then follow from the correspondence between quantum semimartingales and their chaos matrix representations.

The proof of essential self-adjointness is in some respects an extension of the proof of essential self-adjointness, on a common core, of the field operators $\Phi(f)$ in quantum statistical mechanics [5, Proposition 5.2.3] (see also [9, §19.3]).

We adopt two approaches to (c). The terms on each side of (4)are replaced by their Duhamel expansions. Using elementary calculus sides are then rearranged and shown to be equal.

Using a more sophisticated method we prove (4) in the special case that M is obtained from a Brownian martingale *via* the Weiner–Ito transformation. Suppose that W is classical Brownian motion and F is a real-valued bounded adapted process in Wiener space and let M be the classical martingale

$$\mathsf{M}_t = \int_0^t \mathsf{F}_s \, dW_s. \tag{7}$$

The Wiener-Ito isomorphism identifies W with $A + A^\dagger$, the Brownian quantum semimartingale, F with an adapted operator-valued process $\hat{F}$ and M may be identified with a quantum semimartingale $\hat{M}$. It follows from the classical Ito formula that the quantum semimartingale $\hat{M}$ satisfies the quantum Ito formula for each $f \in C^{2+}(\mathbb{R})$.

We then show that if $\hat{J}$ is a regular self-adjoint quantum semimartingale $\hat{M} + \hat{J}$ satisfies the quantum Ito formula. The method is to approximate $\hat{M}$ by a sequence $h_n(\hat{M})$ of regular quantum semimartingales. The regular quantum semimartingale $\hat{N}^{(n)} = h_n(\hat{M}) + \hat{J}$ then satisfies the quantum Ito formula. A limiting argument shows that $f(\hat{M})$ satisfies the quantum Ito formula.

This article is rather long even though the ideas contained in it, outlined above, are simple. In our attempt to keep within a functional analytic category we have had to start from scratch in some places.

The infinite matrix approach to operator theory has not proved very popular: it gives nothing new for bounded operators and the correspondence between unbounded operators and chaos matrices is unclear. There seem to be few results in the literature and we have had to prove those we need. Intrinsic conditions must be found for a chaos matrix valued process to be adapted. The cmx quantum stochastic integrals defined in this paper are processes of chaos matrices and a balance sheet must be kept to show that they represent the corresponding Hudson–Parthasarathy processes. Another complication is that although the integrands for quantum semimartingales belong to Lebesgue spaces they do not necessarily belong to the standard Bochner–Lebesgue spaces. Finally the conditions for unbounded operators to be essentially self-adjoint are delicate and must be checked in detail.

To keep the article to its present length we let the time interval be $[0, 1]$, start our processes at 0, consider functions $f(x)$ rather than $f(x, t)$, dispense with the initial space and do not consider the cases of finite and countably infinite multiplicity.

The following is a special case of Corollary 11.3 which overcomes the difficulty with Brownian motion.

Theorem 1.2 *Let $\hat{F}$ in the Bochner–Lebesgue space $L^2([0,1], \mathcal{B}(\mathfrak{H}))$ be an adapted process and let M be the quantum semimartingale*

$$\hat{M}_t = \int_0^t (\hat{F}_s\, dA_s + \hat{F}_s^* dA_s^\dagger).$$

If F_t is the chaos matrix representation $\hat{F}_t$ then $t \mapsto F_t{}^i_j$ belongs to Bochner–Lebesgue space $L^2([0,1], \mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i))$.

The control matrix $\mathcal{K}$ with entries

$$\mathcal{K}^i_j = i^{\frac{1}{2}}\|F^j_{i-1}\|_2 + \|F^i_{j-1}\|_2 j^{\frac{1}{2}}$$

represents a symmetric linear transformation $\hat{\mathcal{K}}$ in ℓ^2.

If ℓ_{00} consists of analytic vectors for $\hat{\mathcal{K}}$ with radius of convergence greater than fixed ϵ then

(i) *$\hat{M}$ is a process of essentially self-adjoint operators;*

(ii) *If $f \in C^{2+}(\mathbb{R})$ then*

$$f(\hat{M}_t) = f(0) + \int_0^t \Big(Df(\hat{M}_s)(d\hat{M}_s) + D_I^2 f(\hat{M}_s)(d\hat{M}_s, d\hat{M}_s)\Big)$$

When $F = I$ this theorem implies the Ito formula for classical Brownian motion.

The article is organised as follows. Section 2 is a general discussion of chaos matrices and the operators they represent. Section 3 recalls the definitions and some properties of the basic processes occurring in the theories of Hudson–Parsatharathy and Lindsay and shows that they may be represented as processes of chaos matrices. Adapted chaos matrix valued processes are characterised in Section 4.

The cmx quantum stochastic integral $M_t(E, F, G, H)$ of a quadruple of chaos matrix valued processes (E, F, G, H) is defined in Section 6. The integrands (E, F, G, H) are not required to be adapted. Bounds are found for the entries in M_t. When (E, F, G, H) are adapted processes of bounded chaos matrices it is shown that M_t represents the Hudson-Parthasarathy quantum stochastic integral of the corresponding operator processes. This shows that the two definitions of quantum stochastic integral are consistent.

Section 7 is devoted to properties of scalar matrices. In Section 8 the integrands are allowed to be unbounded processes and sufficient conditions are found for a cmx quantum stochastic integrals to represent the corresponding Hudson–Parthasarathy quantum stochastic integral.

The remainder of the article is devoted to the quantum Ito formula. A product Ito formula is proved for adapted chaos matrix valued processes in Section 9, a quantum Duhamel formula is proved in Section 10 and the quantum Ito formula in Section 11.

We give, without proof, in Section 12 a quantum Stratonovich formula for regular quantum semimartingales and pose the problem of generalising it to the irregular case.

In Section 13 it is shown that a classical Brownian quantum semimartingale perturbed by a regular quantum semimartingale still satisfies the quantum Ito formula. The methods of this section seem to work only when the perturbed processes are classical.

The Duhamel expansion is reviewed in Section 14 and used in Section 15 to show that the set of essentially self-adjoint quantum semimartingales which satisfy the quantum Duhamel formula is closed under perturbations by regular self-adjoint quantum semimartingales.

In Section 16 the article concludes with five open problems.

2 Chaos Matrices.

A *chaos decomposition* of a Hilbert space $\mathcal{H}$ is a decomposition of $\mathcal{H}$ into the direct sum

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^j \oplus \cdots$$

of Hilbert spaces $\mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^2, \ldots$. The Hilbert space $\mathcal{H}^j$ is called the jth *chaos* and π^j denotes the orthogonal projection onto $\mathcal{H}^j$. Each $\psi \in \mathcal{H}$ is written as a column vector $\psi = (\psi^0, \psi^1, \cdots, \psi^j, \cdots)^t$ called the *chaos representation* of ψ.

Let $\mathcal{H}_{00} = \{\psi \in \mathcal{H} : \psi^j = 0$ whenever $j > N = N(\psi) \in \mathbb{N}\}$, the dense subspace of vectors with finite chaos representations.

A *chaos matrix* (for the above decomposition of $\mathcal{H}$) is a matrix T with *entries* $T^i_j \in \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)$, $i, j = 0, 1, 2, \ldots$. We often use 'cmx' as shorthand for 'chaos matrix'. For example, 'representation' is short for 'chaos matrix representation'.

The *adjoint* of T is $T^* \stackrel{\text{def}}{=\!=} [(T_i^j)^*]$. If $T = T^*$ then T is a *symmetric* chaos matrix.

A chaos matrix T is $(2k+1)$-*diagonal* if $T_j^i = 0$ whenever $|i-j| > k$.

If S and T are chaos matrices and, for all $i, j = 0, 1, \ldots$, the series $\sum_{\nu=0}^{\infty} S_\nu^i T_j^\nu$ is convergent in the strong operator topology to U_j^i in $\mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)$ then the chaos matrix $U = [U_j^i]$ is the *product* ST of S and T. This product will exist if, for example, either of S and T is $(2k+1)$-diagonal for some $k \in \mathbb{N}$.

The existence of ST does not necessarily imply the existence of T^*S^*.

Lemma 2.1 *Let S and T be chaos matrices. If ST and T^*S^* both exist then $(ST)^* = T^*S^*$.*

Proof. If $\phi \in \mathcal{H}^i$ and $\psi \in \mathcal{H}^j$ then

$$\langle ST\psi, \phi \rangle = \sum_{\nu=0}^{\infty} \left\langle S_\nu^i T_j^\nu \psi, \phi \right\rangle = \sum_{\nu=0}^{\infty} \left\langle \psi, (T^*)_\nu^j (S^*)_i^\nu \phi \right\rangle = \left\langle \psi, T^*S^*\phi \right\rangle . \blacksquare$$

Let $\mathcal{L}(\mathcal{H})$ denote the set of linear transformations in $\mathcal{H}$. To each chaos matrix T corresponds $\tilde{T}$ in $\mathcal{L}(\mathcal{H})$ as follows. Let

$$\mathfrak{D}(T) = \left\{ \psi \in \mathcal{H} : \begin{array}{c} \sum_{j=0}^{\infty} T_j^i \psi^j \text{ is convergent to } \eta^i(\psi) \in \mathcal{H}^i \text{ for all } i \\ \text{and} \\ \eta(\psi) = (\eta^0(\psi), \eta^1(\psi), \eta^2(\psi), \ldots)^t \text{ belongs to } \mathcal{H} \end{array} \right\}.$$

Define $\tilde{T}$ by letting $\mathcal{D}(\tilde{T}) = \mathfrak{D}(T)$ and putting

$$\tilde{T}\psi = \eta(\psi) \qquad \text{whenever } \psi \in \mathfrak{D}(T)$$

Care must be taken at this point. If T is a chaos matrix and the product chaos matrix T^2 exists it is not necessarily true that

$$\mathfrak{D}(T^2) = \mathcal{D}(\tilde{T}^2) = \{\psi \in \mathcal{D}(\tilde{T}) : \tilde{T}\psi \in \mathcal{D}(\tilde{T})\}.$$

It may be that $\psi \in \mathfrak{D}(T^2)$ yet $\psi \notin \mathfrak{D}(T)$.

The C^∞*-vectors* of a chaos matrix T are

$$C^\infty(T) = \begin{cases} \bigcap_{k=0}^\infty \mathfrak{D}(T^k) & \text{if } T^k \text{ exists for } k = 1,2,\ldots, \\ \emptyset & \text{otherwise.} \end{cases}$$

A vector $\varphi \in \mathfrak{H}$ is *analytic* for T if $\varphi \in C^\infty(T)$ and the complex power series

$$\sum_{k=0}^{\infty} \frac{\|T^k\varphi\|}{k!} z^k \tag{8}$$

has non-zero radius of convergence $r(\varphi) = r(\varphi, T)$. Then $\mathcal{A}(T)$ is the vector space of analytic vectors of T and $\mathcal{A}_r(T) = \{\varphi \in \mathcal{A}(T) : r(\varphi) \geq r\}$ for $0 < r \leq \infty$.

Let T be a chaos matrix and let $\hat{T}$ belong to $\mathcal{L}(\mathcal{H})$. Then T is a *chaos matrix representation* of $\hat{T}$ if

$$\text{(i)} \quad \mathcal{D}(\hat{T}) \subset \mathfrak{D}(T); \qquad \text{(ii)} \quad T\varphi = \hat{T}\varphi \text{ whenever } \varphi \in \mathcal{D}(\hat{T}).$$

Thus T is a cmx representation of $\hat{T}$ if and only if $\hat{T}$ is an extension of $\tilde{T}$. We shall not consider operators with more than one chaos-matrix representation.

The number operator $\hat{N}$ in $\mathcal{H}$ has unique diagonal chaos-matrix representation

$$N = \text{diag}[0, I_1, 2I_2 \ldots, jI_j \ldots],$$

where I_j is the identity transformation in $\mathcal{H}^j$. Clearly $\mathfrak{D}(N) = \{\psi \in \mathcal{H} : (0, \psi^1, 2\psi^2, \ldots, j\psi^j, \ldots)^t \in \mathcal{H}\} = \mathcal{D}(\tilde{N})$ and $\tilde{N} = \hat{N}$.

A chaos matrix T is *bounded* if $\mathfrak{D}(T) = \mathcal{H}$ and $\tilde{T}$ is bounded. Let $\mathcal{B}_{\text{cmx}}(\mathcal{H})$ be the Banach space

$$\mathcal{B}_{\text{cmx}}(\mathcal{H}) = \{T : \ T \text{ is a bounded chaos matrix for } \mathcal{H}\}$$

with $\|T\| \stackrel{\text{def}}{=\!=} \|\tilde{T}\|$ whenever $T \in \mathcal{B}_{\text{cmx}}(\mathcal{H})$. The completeness of $\mathcal{B}_{\text{cmx}}(\mathcal{H})$ follows from part (iii) of the following theorem.

Theorem 2.2 (i) *Each* $\hat{T} \in \mathcal{B}(\mathcal{H})$ *has a unique cmx representation* T *whose entries are given by*

$$T^i_j\psi^j = (\hat{T}\psi^j)^i \qquad \psi^j \in \mathcal{H}^j \tag{9}$$

Moreover $\left\|T^i_j\right\| \leq \|T\|$ *for all* $i, j = 0, 1, 2, \ldots$.

(ii) *A chaos matrix T is bounded if and only if $\mathfrak{D}(T) = \mathcal{H}$.*

(iii) *If T is a chaos matrix with $\mathcal{H}_{00} \subset \mathfrak{D}(T)$ and $\tilde{T}$ is bounded then T is bounded.*

(iv) *With the product and adjoint operations defined above $\mathcal{B}_{\mathrm{cmx}}(\mathcal{H})$ is a von Neumann algebra. The mapping $T \mapsto \tilde{T}$ is an isometric algebraic $*$-isomorphism from $\mathcal{B}_{\mathrm{cmx}}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})$.*

Proof. (i) It follows from the definition that any cmx representation of $\hat{T}$ satisfies (9). If $\psi^j \in \mathcal{H}^j$ then $\|(\hat{T}\psi^j)^i\| \leq \|\hat{T}\psi^j\| \leq \|\hat{T}\| \cdot \|\psi^j\|$ so that (9) defines $T^i_j \in \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)$. Therefore the chaos matrix T is well defined by (9).

$\hat{T}$ is bounded and if $\psi \in \mathcal{H}$ then $\sum_{j=0}^{\infty} T^j_i \psi^j = \sum_{j=0}^{\infty} (\hat{T}\psi^j)^i = (\hat{T}\psi)^i = \eta^i$ where $\eta = \hat{T}\psi$. Therefore $\mathfrak{D}(T) = \mathcal{H}$ and T uniquely represents $\hat{T}$.

(ii) If T is bounded then $\mathfrak{D}(T) = \mathcal{H}$ by definition.

Suppose, conversely, that $\mathfrak{D}(T) = \mathcal{H}$. Define the linear transformation $\tilde{T}^i : \mathcal{H} \to \mathcal{H}^i$ by

$$\tilde{T}^i \psi = \sum_{j=0}^{\infty} T^i_j \psi^j = \sum_{j=0}^{\infty} (T^i_j \circ \pi^j)\psi = \eta^i$$

whenever $\psi \in \mathcal{H}$. Since $T^i_j \circ \pi^j$ is bounded it follows from the uniform boundedness principle that $\tilde{T}^i \in \mathcal{B}(\mathcal{H}, \mathcal{H}^i)$. It follows from the definition of $\mathfrak{D}(T)$ that $\sum_{i=0}^{\infty} \tilde{T}^i \psi$ is convergent to $\tilde{T}\psi$ for each $\psi \in \mathcal{H}$ and by the uniform boundedness principle $\tilde{T} \in \mathcal{B}(\mathcal{H})$.

(iii) The bounded linear operator $\tilde{T}$ has a unique extension $\hat{T} \in \mathcal{B}(\mathcal{H})$. Since $\mathcal{H}_{00} \subset \mathfrak{D}(T)$ the operator T^i_j is defined by (9). Therefore T represents $\hat{T}$ and $\mathfrak{D}(T) = \mathcal{H}$.

(iv) If $S, T \in \mathcal{B}_{\mathrm{cmx}}(\mathcal{H})$ and $\psi^j \in \mathcal{H}^j$ then $(T^0_j \psi^j, T^1_j \psi^j, \cdots, T^n_j \psi^j, \cdots)^t = \tilde{T}\psi^j \in \mathcal{H}$. Since $\mathfrak{D}(S) = \mathcal{H}$

$$(\tilde{S}(\tilde{T}\psi^j))^i \;=\; \sum_{\nu=0}^{\infty} S^i_\nu T^\nu_j \psi^j. \tag{10}$$

Therefore the series converges for each $\psi^j \in \mathcal{H}^j$. By the uniform boundedness theorem $\sum_{\nu=0}^{\infty} S^i_\nu T^\nu_j$ is convergent in the strong operator topology to U^i_j in $\mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)$. Therefore the product chaos matrix ST exists.

It follows from (10) that, for $\varphi^i \in \mathcal{H}^i$ and $\psi^j \in \mathcal{H}^j$,

$$\left\langle \tilde{S}\tilde{T}\psi^j, \varphi^i \right\rangle = \left\langle U^i_j \psi^j, \varphi^i \right\rangle = \left\langle (ST)^i_j \psi^j, \varphi^i \right\rangle .$$

Therefore ST is the chaos matrix of $\tilde{S}\tilde{T}$ and $\widetilde{ST} = \tilde{S}\tilde{T}$.

If $\psi^j \in \mathcal{H}^j$ and $\varphi^i \in \mathcal{H}^i$ then

$$\left\langle \tilde{T}\psi^j, \varphi^i \right\rangle = \left\langle T^i_j \psi^j, \varphi^i \right\rangle = \left\langle \psi^j, (T^i_j)^* \varphi^i \right\rangle = \left\langle \psi^j, (T^*)^j_i \varphi^i \right\rangle = \left\langle \psi^j, \widetilde{T^*} \varphi^i \right\rangle .$$

It follows from (i) that T^* is the cmx representation of $(\tilde{T})^*$ and $\widetilde{(T^*)} = (\tilde{T})^*$. The transformation $T \mapsto \tilde{T}$ is clearly linear and is therefore an isometric $*$-isomorphism from $\mathcal{B}_{\text{cmx}}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})$. ∎

Let $\mathcal{B}(\mathcal{H})_{\text{so}}$ denote $\mathcal{B}(\mathcal{H})$ with the strong operator topology and let $\mathcal{B}_{\text{cmx}}(\mathcal{H})_{\text{so}}$ denote $\mathcal{B}_{\text{cmx}}(\mathcal{H})$ with the quotient topology induced by the $*$-isomorphism $T \mapsto \tilde{T}$.

An *operator process* is a family $\hat{T} = \{\hat{T}_t : t \in [0, 1]\}$ of linear operators. If $\hat{T}_t$ is bounded for almost all $t \in [0, 1]$ then $\hat{T}$ is *regular*. Two operator processes $\hat{T}$ and $\hat{T}'$ are identified if $\hat{T}_t = \hat{T}'_t$ for almost all $t \in [0, 1]$.

A *chaos matrix process* (*cmx process*) is a family $T = \{T_t : t \in [0, 1]\}$ of chaos matrices. If T_t is a bounded chaos matrix for almost all $t \in [0, 1]$ then T is *regular*. Two cmx processes T and T' are identified if $T_t = T'_t$ for almost all $t \in [0, 1]$.

A cmx process T is a *cmx representation* of an operator process $\hat{T}$ if T_t is a cmx representation of $\hat{T}_t$ for almost all $t \in [0, 1]$.

The integrands of the quantum semimartingales considered by Attal and Meyer are regular processes $\hat{F} = \{\hat{F}_t : t \in [0, 1]\}$ where $\hat{F}_t$ is a bounded operator in the Bose-Fock space $\mathfrak{H}$ for each $t \in [0, 1]$. Since $\mathcal{B}(\mathfrak{H})$ is non-separable the measurability conditions on these integrands, defined with respect to the strong operator topology on $\mathcal{B}(\mathfrak{H})$, are too weak to ensure that $\hat{F}$ belongs to one of the Bochner–Lebesgue spaces $L^p([0, 1], \mathcal{B}(\mathfrak{H}))$. Nor are they strong enough to ensure that $\hat{F}$ is Pettis integrable. The integrand $\hat{F}$ belongs instead to one of the following Lebesgue spaces.

If $1 \leq p \leq \infty$ let $L^p([0,1], \mathcal{B}(\mathcal{H})_{\mathrm{so}})$ the normed vector space of processes $\hat{F} : [0,1] \to \mathcal{B}(\mathcal{H})$ such that

(i) $t \mapsto \hat{F}_t \xi$ is measurable for each ξ in some dense subset of $\mathcal{H}$;

(ii) $t \mapsto \left\| \hat{F}_t \right\|$ belongs to $L^p([0,1])$,

with norm, $\|\cdot\|_p$, given by

$$\left\| \hat{F} \right\|_p = \begin{cases} \left(\int_0^1 \left\| \hat{F}_t \right\|^p \, dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup\limits_{t \in [0,1]} \left\| \hat{F}_t \right\|, & p = \infty. \end{cases}$$

Suppose ξ and $\xi^{(n)}$ in $\mathcal{H}$ are such that $\left\| \xi - \xi^{(n)} \right\| < 1/2^n$. Then $t \mapsto \hat{F}_t \xi$ and $t \mapsto \hat{F}_t \xi^{(n)}$ belong to the Bochner–Lebesgue space $L^p([0,1], \mathcal{H})$ and $\left\| \hat{F}_{\bullet}\, \xi - \hat{F}_{\bullet}\, \xi^{(n)} \right\| < \left\| \hat{F} \right\|_p / 2^n$. Therefore $\hat{\hat{F}}_t \xi^{(n)}$ converges to $\hat{\hat{F}}_t \xi$ for almost all t in $[0,1]$ [25, Theorem 3.11]. This means that (i) may be replaced by the condition

(i)$'$ $t \mapsto \hat{F}_t \xi$ is measurable for each ξ in $\mathcal{H}$.

Condition (i) is used rather than (i)$'$ because the measurability condition on an integrand, $\hat{F}$, of a quantum semimartingale is that $t \mapsto \hat{F}_t e(f)$ be measurable whenever the exponential vector $e(f)$ belongs to a given total set of exponential vectors in $\mathfrak{H}$.

A process $\hat{F}$ satisfying (i)$'$ is *strongly measurable.* The composition of strongly measurable processes is strongly measurable.

Corollary 2.3 *If $\hat{F}, \hat{G} : [0,1] \to \mathcal{B}(\mathcal{H}))$ are strongly measurable then so is $\hat{F} \circ \hat{G} : [0,1] \to \mathcal{B}(\mathcal{H})$, where $(\hat{F} \circ \hat{G})_t = \hat{F}_t \circ \hat{G}_t$.*

Proof. If $\xi \in \mathcal{H}$ there exists a sequence $\varphi^{(n)}$ of $\mathcal{H}$-valued step functions such that $\hat{G}_t \xi = \lim_{n \to \infty} \varphi_t^{(n)}$ almost everywhere. But $t \mapsto \hat{F}_t \otimes \hat{G} v_t^{(n)}$ is measurable and converges almost everywhere to $\hat{F}_t \otimes \hat{G}_t \xi$ so that $t \mapsto \hat{F}_t \otimes \hat{G}_t \xi$ is measurable. ∎

Lemma 2.4 *If $1 \le p \le \infty$ then $L^p([0,1], \mathcal{B}(\mathcal{H})_{\rm so})$ is a Banach space.*

Proof. We follow the line of argument in [25]. Suppose $1 \le p < \infty$ and let $\hat{F}^{(n)}$ be a Cauchy sequence in $1 \le p \le \infty$. By extracting a subsequence it may be assumed that $\left\|\hat{F}^{(m)} - \hat{F}^{(n)}\right\| < 1/2^n$ whenever $m > n$. This implies that $\hat{F}_t^{(n)}$ is norm convergent in $\mathcal{B}(\mathcal{H})$ to $\hat{F}_t$ for almost all t in $[0,1]$. If $\xi \in \mathcal{H}$ then $\hat{F}_t^{(n)}\xi$ is almost everywhere convergent to $\hat{F}_t\xi$ and $t \mapsto \hat{F}_t\xi$ is measurable.

By Fatou's lemma

$$\int_0^1 \left\|\hat{F}_t - \hat{F}_t^{(n)}\right\|^p dt \le \liminf_{m\to\infty} \int_0^1 \left\|\hat{F}_t^{(m)} - \hat{F}_t^{(n)}\right\| dt \le \frac{1}{2^n p}, \tag{11}$$

and $t \mapsto \left\|\hat{F}_t - \hat{F}_t^{(n)}\right\|$ belongs to $L^p[0,1]$. Thus $\hat{F} - \hat{F}^{(n)}$ and therefore $\hat{F}$ belong to $L^p([0,1], \mathcal{B}(\mathcal{H})_{\rm so})$. It follows from (11) that $\hat{F}^{(n)}$ is norm convergent to $\hat{F}$ so that $L^p([0,1], \mathcal{B}(\mathcal{H})_{\rm so})$ is a Banach space.

A straightforward argument shows that $L^\infty([0,1], \mathcal{B}(\mathcal{H})_{\rm so})$ is also a Banach space. ∎

We now define the integral of $\hat{F} \in L^p([0,1], \mathcal{B}(\mathcal{H})_{\rm so})$ for $1 \le p \le \infty$. It follows from (ii) that $t \mapsto \left\|\hat{F}_t\xi\right\|$ belongs to $L^p[0,1]$ and from (i)$'$ that $t \mapsto \hat{F}_t\xi$ belongs to the Bochner–Lebesgue space $L^p([0,1], \mathcal{H})$ for each $\xi \in \mathcal{H}$ [6, II.2. Theorem 2]. Define the indefinite integral, $I_\bullet(\hat{F})$, of $\hat{F}$ by putting

$$I_t(\hat{F})\xi \equiv \left(\int_0^t \hat{F}_s\, ds\right)\xi \stackrel{\rm def}{=\!=} \int_0^t \hat{F}_s\xi\, ds, \qquad \xi \in \mathcal{H}, \quad t \in [0,1].$$

If $\eta \in \mathcal{H}$ then

$$\begin{aligned} |\left\langle I_t(\hat{F})\xi, \eta\right\rangle| = \left|\int_0^t \left\langle \hat{F}_s\xi, \eta\right\rangle ds\right| &\le \int_0^t \left\|\hat{F}_s\right\| ds\, \|\xi\|\ \|\eta\| \\ &\le t^{\frac{1}{q}} \left\|\hat{F}\right\|_p \|\xi\|\ \|\eta\|. \end{aligned}$$

This shows that the indefinite integral $I_t(\hat{F})$ belongs to $\mathcal{B}(\mathcal{H})$ with $\left\|I_t(\hat{F})\right\| \le t^{1/q}\left\|\hat{F}\right\|_p$.

The vector space of cmx processes is denoted $\mathcal{L}_{\text{cmx}}(\mathcal{H})$. Define the Frechet spaces

$$\begin{array}{rcl} \mathcal{L}^p_{\text{cmx}}(\mathcal{H})_{\text{so}} &=& \{T \in \mathcal{L}_{\text{cmx}}(\mathcal{H}) : T^i_j \in L^p([0,1], \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)_{\text{so}})\}, \\ \mathcal{L}^p_{\text{cmx}}(\mathcal{H}) &=& \{T \in \mathcal{L}_{\text{cmx}}(\mathcal{H}) : T^i_j \in L^p([0,1], \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i))\}, \end{array} \qquad 1 \le p \le \infty,$$

with the topology defined by the seminorms $X \mapsto \left\|X^i_j\right\|_p$ $i, j = 0, 1, \ldots$ where $\|\cdot\|_p$ is the norm on $L^p([0,1], \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)_{\text{so}})$ defined above.

Thus a sequence $S^{(n)}$ in $\mathcal{L}^p_{\text{cmx}}(\mathcal{H})$ is *convergent* to S in $\mathcal{L}^p_{\text{cmx}}(\mathcal{H})$ if $(S^{(n)})^i_j$ is convergent in $L^p([0,1], \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i))$ to S^i_j for all i, j. A series $S^{(n)}$ is *summable* (*absolutely summable*) if the series $(S^{(n)})^i_j$ is summable (absolutely summable) for each i, j.

Suppose $p, q, r \in [1, \infty]$ with $1/p + 1/q = 1/r$. If $X \in \mathcal{L}^p_{\text{cmx}}(\mathcal{H})$ and $Y \in \mathcal{L}^q_{\text{cmx}}(\mathcal{H})$ then the formula

$$(X^i_\nu Y^\nu_j)_t = X_t{}^i_\nu Y_t{}^\nu_j$$

defines, for each $\nu \in \mathbb{N}$, a process $X^i_\nu Y^\nu_j \in L^r([0,1], \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i))$. If $\sum_{\nu=0}^\infty X^i_\nu Y^\nu_j$ converges to Z^i_j in $L^r([0,1], \mathcal{B}_{\text{so}}(\mathcal{H}^j, \mathcal{H}^i))$ for each $i, j \in \mathbb{N}$ the cmx process Z is the *product* of X and Y. We put $Z = XY$.

If $1 \le p \le \infty$ define $L^p([0,1], \mathcal{B}_{\text{cmx}}(\mathcal{H})_{\text{so}})$ to be the normed vector space of cmx processes F such that

(i) $t \mapsto F_t{}^i_j \xi^j$ is measurable for each ξ^j in a dense subset of $\mathcal{H}^j$;

(ii) $t \mapsto \|F_t\|$ belongs to $L^p[0,1]$.

As in the case of processes of bounded operator condition (i) is equivalent to

(i)′ $t \mapsto F_t{}^i_j \xi^j$ is measurable for each ξ^j in $\mathcal{H}^j$.

We note the following inclusions between these spaces.

Proposition 2.5

(i) $L^p([0,1], \mathcal{B}_{\text{cmx}}(\mathcal{H})_{\text{so}})$ *is a closed subspace of* $\mathcal{L}^p_{\text{cmx}}(\mathcal{H})_{\text{so}}$.

(ii) $L^p([0,1], \mathcal{B}_{\text{cmx}}(\mathcal{H}))$ *is a closed subspace of* $\mathcal{L}^p_{\text{cmx}}(\mathcal{H})$.

(iii) $L^p([0,1],\mathcal{B}_{\mathrm{cmx}}(\mathcal{H}))$ *is a closed subspace of* $L^p([0,1],\mathcal{B}_{\mathrm{cmx}}(\mathcal{H})_{\mathrm{so}})$.

(iv) $L^p([0,1],\mathcal{B}(\mathcal{H}))$ *is a closed subspace of* $L^p([0,1],\mathcal{B}(\mathcal{H})_{\mathrm{so}})$

We also have the following correspondences between regular operator processes and regular cmx processes.

If $\hat{X} \in L^p([0,1],\mathcal{B}(\mathcal{H})_{\mathrm{so}})$ define the cmx process X by letting X_t be the cmx representation of $\hat{X}_t$ whenever $t \in [0,1]$.

If $X \in L^p([0,1],\mathcal{B}_{\mathrm{cmx}}(\mathcal{H})_{\mathrm{so}})$ define the operator process $\tilde{X}$ by putting $\tilde{X}_t = \widetilde{(X_t)}$ whenever $t \in [0,1]$.

Proposition 2.6 *If* $1 \le p \le \infty$ *then the mapping* $\hat{X} \mapsto X$ *is an isometric isomorphism from*

(i) $L^p([0,1],\mathcal{B}(\mathcal{H})_{\mathrm{so}})$ *onto* $L^p([0,1],\mathcal{B}_{\mathrm{cmx}}(\mathcal{H})_{\mathrm{so}})$*;*

(ii) $L^p([0,1],\mathcal{B}(\mathcal{H}))$ *onto* $L^p([0,1],\mathcal{B}_{\mathrm{cmx}}(\mathcal{H}))$*.*

Moreover $\tilde{X} = \hat{X}$*.*

The process X is the *cmx representation* of $\hat{X}$.

The relationship between unbounded chaos matrices and the operators they represent is not straightforward. The chaos matrices are a vector space for entrywise addition and scalar multiplication while $\mathcal{L}(\mathcal{H})$ is not a vector space. There are non-trivial chaos matrices which represent only the trivial operator with domain $\{0\}$. Although a chaos matrices always represents a linear transformation not every linear transformation has a chaos matrix representation. Nor is the correspondence between a chaos matrix T and the operator $\tilde{T}$ always one to one.

However under some conditions symmetric chaos matrices represent unbounded essentially self-adjoint linear transformations.

If X is in $\mathcal{L}^p_{\mathrm{cmx}}(\mathcal{H})_{\mathrm{so}}$ then $\int_0^t X_s\,ds$ is the cmx process with entries

$$\int_0^t X_s{}^i_j\,ds.$$

Theorem 2.7 *Let* X *be a process in* $L^p([0,1],\mathcal{B}_{\mathrm{cmx}}(\mathcal{H})_{\mathrm{so}})$ *where* $1 \le p \le \infty$*. If* $1/p + 1/q = 1$ *and* $i,j = 0,1,2,\ldots$ *then, for* $0 \le t \le 1$*,*

(i) $X^i_j \in L^p([0,1], \mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)_{\mathrm{so}})$ *with* $\left\|X^i_j\right\|_p \le \|X\|_p$.

(ii) *The linear transformation* $Y_t{}^i_j$ *defined by the formula*

$$Y_t{}^i_j\psi = \int_0^t X_u{}^i_j\psi\, du, \qquad \psi \in \mathcal{H}^j,$$

belongs to $\mathcal{B}(\mathcal{H}^j, \mathcal{H}^i)$ *and* $\|Y_t{}^i_j\| \le t^{1/q}\left\|X^i_j\right\|_p \le t^{1/q}\|X\|_p$.

(iii) *The series* $\sum_{j=0}^\infty Y_t{}^i_j\psi^j$ *is convergent for each* $\psi \in \mathcal{H}$.

(iv) *The chaos matrix* Y_t *with entries* $Y_t{}^i_j$ *is bounded with* $\|Y_t\| \le t^{1/q}\|X\|_p$.

(v) *If* $\psi \in \mathcal{H}$ *then*

$$\tilde{Y}_t\psi = \int_0^t \tilde{X}_u\psi\, du.$$

Proof. (i) If $\psi \in \mathcal{H}^j$ then $X_u{}^i_j\psi = \pi^i\tilde{X}_u\psi$. Since π^i is continuous and $u \mapsto \tilde{X}_u\psi$ is measurable it follows that $u \mapsto X_u{}^i_j\psi$ is measurable. The norm bound follows from the fact that $\left\|X_u{}^i_j\right\| \le \|X_u\|$ for all $u \in [0,1]$.

(ii) $$\left\|\int_0^t X_u{}^i_j\psi\, du\right\| \le \int_0^t \left\|X_u{}^i_j\psi\right\| du \le \int_0^t \left\|X_u{}^i_j\right\| du\, \|\psi\| \le t^{\frac{1}{q}}\|X\|_p\|\psi\|$$

(iii) If $\varphi \in \mathcal{H}_{00}$ then, interchanging finite sums and integrals,

$$\begin{aligned}
\left\|\sum_{j=0}^\infty Y_t{}^i_j\varphi^j\right\| &= \left\|\int_0^t \sum_{j=0}^\infty X_u{}^i_j\varphi^j\, du\right\| \\
&= \left\|\int_0^t (\tilde{X}_u\varphi)^i\, du\right\| \\
&= \left\|\left(\int_0^t \tilde{X}_u\varphi\, du\right)^i\right\| \\
&\le \int_0^t \left\|\tilde{X}_u\right\| du\, \|\varphi\| \\
&\le t^{\frac{1}{q}}\|X\|_p\|\varphi\|.
\end{aligned}$$

Thus the series $\sum_{j=0}^\infty Y_t{}^i_j\psi^j$ satisfies the Cauchy condition for convergence.

(iv) If $\psi \in \mathcal{H}$ and $\psi^{(n)} = (\psi^0, \psi^1, \ldots, \psi^n, 0, \ldots)$ then

$$\left\| \sum_{j=0}^{n} X_u{}^i_j \psi^j \right\| = \left\| (\tilde{X}_u \psi^{(n)})^i \right\| \le \left\| (\tilde{X}_u \right\| \cdot \|\psi^{(n)}\| \le \left\| (\tilde{X}_u \right\| \cdot \|\psi\| .$$

By the dominated convergence theorem

$$\int_0^t \left(\sum_{j=0}^{\infty} X_u{}^i_j \psi^j \right) du = \sum_{j=0}^{\infty} \int_0^t X_u{}^i_j \psi^j \, du = \sum_{j=0}^{\infty} Y_t{}^i_j \psi$$

converges to η^i in $\mathcal{H}^i$. If $\phi \in \mathcal{H}$ then

$$\begin{aligned} \sum_{i=0}^{\infty} | \left\langle \eta^i, \phi^i \right\rangle | &= \sum_{i=0}^{\infty} \left| \int \left\langle X_u{}^i_j \psi^j, \phi^i \right\rangle du \right| \\ &= \sum_{i=0}^{\infty} \left| \int \left\langle (\tilde{X}_u \psi)^i, \phi^i \right\rangle du \right| \\ &\le \|\psi\| \cdot \|\phi\| \int_0^t \|X_u\| \, du, \end{aligned} \tag{12}$$

where the inequality follows from the dominated convergence theorem and the Cauchy–Schwarz inequality

$$\sum_{i=0}^{\infty} | \left\langle (\tilde{X}_u \psi)^i, \phi \right\rangle | \le \left\| \tilde{X}_u \psi \right\| \cdot \|\phi\| \le \|X_u\| \cdot \|\psi\| \cdot \|\phi\| .$$

This shows that $\mathfrak{D}(Y_t) = \mathcal{H}$ and Y_t is bounded with $\|Y_t\| \le t^{1/q} \|X\|_p$.

(v) It follows from (12) and the dominated convergence theorem that

$$\left\langle \tilde{Y}_t \psi, \phi \right\rangle = \int_0^t \left\langle \tilde{X}_u \psi, \phi \right\rangle du = \left\langle \int_0^t \tilde{X}_u \psi \, du, \phi \right\rangle \blacksquare$$

If $\mathcal{K}$ is a complex Hilbert space with chaos decomposition $\mathcal{K} = \bigoplus_{j=0}^{\infty} \mathcal{K}^j$ the forgoing analysis above may be carried out with $\mathcal{B}(\mathcal{H})$, $\mathcal{B}_{\mathrm{cmx}}(\mathcal{H})$,... replaced by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{B}_{\mathrm{cmx}}(\mathcal{H}, \mathcal{K})$,....

We shall only consider two chaos decompositions in this article:

1. *Complex Hilbert sequence space* ℓ^2. This is the direct sum of one-dimensional Hilbert spaces. In this case chaos matrices are called *scalar matrices*.

The scalar matrices have a partial ordering: $\mathcal{K} \prec \mathcal{V}$ if $\mathcal{K}^i_j \leq \mathcal{V}^i_j$ for all i, j.

2. *Boson Fock space*. Let $\mathfrak{H}_{t]}{}^j = L^2_{\text{sym}}([0,t]^j)$ be the Hilbert space of totally symmetric square-integrable functions on $[0,t]^j$ and let

$$\mathfrak{H}_{t]} \stackrel{\text{def}}{=\!=} \mathfrak{H}^0 \oplus \mathfrak{H}_{t]}{}^1 \oplus \cdots \oplus \mathfrak{H}_{t]}{}^j \oplus \cdots. \tag{13}$$

Then $\mathfrak{H}_t$ is the Boson Fock space over $L^2[0,t]$ often denoted $\mathfrak{F}_+(L^2[0,t])$. This is one of several equivalent representations of $\mathfrak{F}_+(L^2[0,t])$. The chaos decomposition (13) is the *Fock decomposition* and will be used throughout this article. If $t = 1$ put $\mathfrak{h} = \mathfrak{H}_{1]}$ $\mathfrak{H}^j = \mathfrak{H}_{1]}{}^j$

If $g \in L^2[0,1]$ let $e^j(g) = (j!)^{-1/2} g \otimes \cdots \otimes g \in \mathfrak{H}^j$. It is an easy consequence of the Stone-Weierstrass theorem that $\mathcal{E}^j$, the linear span of $\{e^j(g) : g \in L^2[0,1]\}$ is dense in $\mathfrak{H}^j$. The *exponential vector* $e(g) \in \mathfrak{H}$ is defined by

$$e(g) \stackrel{\text{def}}{=\!=} (1, e^1(g), e^2(g), \cdots, e^j(g), \cdots)^t = (1, g, \frac{g^{\otimes 2}}{(2!)^{1/2}}, \cdots, \frac{g^{\otimes j}}{(j!)^{1/2}}, \cdots)^t.$$

The *exponential domain*, $\mathcal{E}_S$, $S \subset L^2[0,1]$ is the linear span of $\{e(g) : g \in L\}$. When $S = L^2[0,1]$ then we put $\mathcal{E}_S = \mathcal{E}$.

The Bose–Fock spaces $\mathfrak{H}_{(t} = \bigoplus_{j=0}^{\infty} \mathfrak{H}_{(t}{}^j$, with $\mathfrak{H}_{(t}{}^j = L^2_{\text{sym}}((0,1]^j)$ are similarly defined.

Both $\mathfrak{H}_{t]}$ and $\mathfrak{H}_{(t}$ are subspaces of $\mathfrak{H}$ and $\mathfrak{H}$ can be identified with $\mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t}$. There is a unitary map $\mathfrak{U} = \mathfrak{U}_t : \mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t} \to \mathfrak{H}$ such that, whenever $g \in L^2[0,t]$ and $h \in L^2(t,1]$,

$$\mathfrak{U}(e(g) \otimes e(h)) = e(g+h).$$

For each linear transformation $\hat{T}_{t]}$ in $\mathfrak{H}_{t]}$ whose domain contains the exponential domain generated by $L^2[0,t]$ the formula $\hat{T}_t = \mathfrak{U} \circ (\hat{T}_{t]} \otimes \hat{I}_{(t)}) \circ \mathfrak{U}^{-1}$ defines a linear operator in $\mathfrak{H}$, the *ampliation* of $T_{t]}$.

In order to describe ampliation in terms of chaos matrices we briefly review the construction of general Bose-Fock space over a complex Hilbert space H.

Let $H^0 = \mathbb{C}$ and let $H^j = H \otimes \cdots \otimes H$ the j-fold tensor product of H with itself. The *Fock space* of H is

$$\mathfrak{F}(H) = \bigoplus_{j=0}^{\infty} H^j.$$

Let S_j be the group of permutations of $\{1, 2, \ldots, j\}$. There is a unique orthogonal projection P in $\mathfrak{F}(H)$ such that

$$P(f_1 \otimes \cdots \otimes f_j) = (j!)^{-1} \sum_{\sigma \in S_j} f_{\sigma_1} \otimes \cdots \otimes f_{\sigma_j}$$

whenever $f_1, \ldots, f_j \in H$. The *Bose–Fock space* $\mathfrak{F}_+(H)$ is defined by

$$\mathfrak{F}_+(H) = P\mathfrak{F}(H).$$

If $\mathcal{H} = \mathfrak{F}_+(H)$ and $\mathcal{H}^j = PH^j$ then the Fock decomposition

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^j \oplus \cdots$$

is a chaos decomposition of $\mathcal{H}$. The *exponential vector* of $h \in h$ is

$$e(h) = 1 \oplus h \oplus \frac{h^{(2)}}{(2!)^{1/2}} \oplus \cdots \oplus \frac{h^{(j)}}{(j!)^{1/2}} \oplus,$$

where $h^{(j)} = h \otimes \cdots \otimes h$, the j-fold tensor product of h with itself. The set $\{e(h) : h \in H\}$ of exponential vectors is total and linearly independent. If $h_1, \ldots, h_j \in H$ then $P(h_1 \otimes \cdots \otimes h_j)$ is in the linear span of $\{h^{(j)} : h \in H\}$. It follows that $\{h^{(j)} : h \in H\}$ is a total subset of $\mathcal{H}^j$.

Suppose $H = H_1 \oplus H_2$ is the direct sum of orthogonal subspaces H_1 and H_2 and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the corresponding Bose–Fock spaces. There is a unique unitary transformation $\mathfrak{U}$ from $\mathcal{H}_1 \otimes \mathcal{H}_2$ onto $\mathcal{H}$ such that

$$\mathfrak{U}(e(h_1) \otimes e(h_2)) = e(h_1 + h_2) \quad \text{whenever } h_1 \in H_1 \text{ and } h_2 \in H_2. \tag{14}$$

For $i = 1, 2$ let $\mathcal{H}_i = \bigoplus_{j=0}^{\infty} \mathcal{H}_i{}^j$ be the Fock decomposition. Then $\mathcal{H}_1{}^{j-r} \otimes \mathcal{H}_2{}^r$ may be regarded as a subspace of $\mathfrak{F}(H)^j$ by putting

$$h_1^{(j-r)} \otimes h_2^{(r)} = \overbrace{h_1 \otimes \cdots \otimes h_1}^{j-r \text{ times}} \otimes \overbrace{h_2 \otimes \cdots \otimes h_2}^{r \text{ times}}$$

whenever $h_i \in H_1$. For $\alpha \in \mathcal{H}_1{}^{j-r}$ and $\beta \in \mathcal{H}_2{}^r$ define $\alpha \otimes_{\mathrm{sym}} \beta = P(\alpha \otimes \beta)$ and put $\mathcal{H}_1{}^{j-r} \otimes_{\mathrm{sym}} \mathcal{H}_2{}^r = P(\mathcal{H}_1{}^{j-r} \otimes \mathcal{H}_2{}^r)$. The vectors $\{h_1{}^{(j-r)} \otimes_{\mathrm{sym}} h_2{}^{(r)} : h_1 \in H_1,\ h_2 \in H_2\}$ are total in $\mathcal{H}_1{}^{j-r} \otimes_{\mathrm{sym}} \mathcal{H}_2{}^r$. The subspaces $\{\mathcal{H}_1{}^{j-r} \otimes_{\mathrm{sym}} \mathcal{H}_2{}^r : 0 \le r \le j\}$ of $\mathcal{H}$ are orthogonal.

Proposition 2.8

(i)
$$\mathcal{H}^j = \bigoplus_{r=0}^{j} \left(\mathcal{H}_1{}^{j-r} \otimes_{\mathrm{sym}} \mathcal{H}_2{}^r\right)$$

and is the closed linear span of the vectors

$$\psi^j = \alpha^j \otimes_{\mathrm{sym}} \beta^0 + \cdots + \alpha^{j-r} \otimes_{\mathrm{sym}} \beta^r \cdots + \alpha^0 \otimes_{\mathrm{sym}} \beta^j, \qquad \alpha^{j-r} \in \mathcal{H}_1{}^{j-r},\ \beta^r \in \mathcal{H}_2{}^r.$$

(ii)
$$\mathfrak{U}(\mathcal{H}_1{}^{j-r} \otimes \mathcal{H}_2{}^r) = \mathcal{H}_1{}^{j-r} \otimes_{\mathrm{sym}} \mathcal{H}_2{}^r.$$

(iii)
$$\mathfrak{U}(\alpha^{j-r} \otimes \beta^r) = \begin{pmatrix} j \\ r \end{pmatrix}^{\frac{1}{2}} \alpha^{j-r} \otimes_{\mathrm{sym}} \beta^r.$$

Proof. If $h_1 \in H_1$, $h_2 \in H_2$ and $a, b \in \mathbb{R}$ then, by (14),

$$\begin{aligned} \mathfrak{U}(e(ah_1) \otimes e(bh_2)) &= \mathfrak{U}\left(\sum_{j=0}^{\infty} \sum_{r=0}^{j} a^{j-r} b^r \frac{h_1^{(j-r)} \otimes h_2^{(r)}}{((j-r)!r!)^{1/2}}\right) \\ &= \sum_{j=0}^{\infty} \frac{(ah_1 + bh_2)^{(j)}}{(j!)^{1/2}}. \end{aligned}$$

Since $P((ah_1 + bh_2)^{(j)}) = (ah_1 + bh_2)^{(j)}$,

$$\begin{aligned} \mathfrak{U}\left(\sum_{r=0}^{j} a^{j-r} b^r \frac{h_1^{(j-r)} \otimes h_2^{(r)}}{((j-r)!r!)^{1/2}}\right) &= \frac{(ah_1 + bh_2)^{(j)}}{(j!)^{1/2}} \qquad (15) \\ &= \sum_{r=0}^{j} a^{j-r} b^r \begin{pmatrix} r \\ j \end{pmatrix} h_1^{(j-r)} \otimes h_2^{(r)} \\ &= \sum_{r=0}^{j} a^{j-r} b^r \begin{pmatrix} r \\ j \end{pmatrix} h_1^{(j-r)} \otimes_{\mathrm{sym}} h_2^{(r)} \qquad (16) \end{aligned}$$

The vectors on the right hand side of Equation (15) are total in $\mathcal{H}^j$ so that (i) follows from (16). Parts (ii) and (iii) follow from equating the coefficients of $a^{j-r} b^r$ in (15) and (16). ∎

A subspace S of $L^2[0,\infty]$ is *admissible* if

(i) $\chi_{[0,t]}g$ belongs to S for all $t \in [0,1]$ whenever $g \in S$;

(ii) $S + iS$ is dense in $L^2[0,1]$,

Hudson and Parthasarathy [11] use for their integrands processes of operators whose domains contain $\mathcal{E}_S$ for some fixed admissible subspace S of $L^2[0,1]$. Fortunately, with these domain restrictions the correspondence between chaos matrix T and linear transformation $\tilde{T}$ is one to one.

Lemma 2.9 *Let $\hat{T}$ be a linear transformation in $\mathcal{H}$ with $\mathcal{E}_S \subset \mathcal{D}(\hat{T})$ for some admissible subspace S of $L^2[0,1]$. If T is a chaos matrix representation of $\hat{T}$ then for each $g \in S$, the sum*

$$\sum_{j=0}^{\infty} T^i_j e^j(g) \tag{17}$$

is absolutely convergent in $\mathfrak{H}^i$ for $i = 0, 1, 2, \ldots$.

Consequently T is the only chaos matrix representation of $\hat{T}$.

Proof. Since S is a complex vector space $e(zg)$ is in $\mathcal{E}_S$ whenever $g \in S$ and $z \in \mathbb{C}$. Now $e^j(zg) = z^j e(g)$ so that the power series $\sum_{j=0}^{\infty} z^j T^i_j e^j(g)$ is convergent, and therefore absolutely convergent, in $\mathfrak{H}^i$ for all $z \in \mathbb{C}$.

If S is a cmx representation of $\hat{T}$ then $\sum_{j=0}^{\infty} z^j T^i_j e^j(g) = \sum_{j=0}^{\infty} z^j S^i_j e^j(g)$ for all $z \in \mathbb{C}$. Therefore $S^i_j e^j(g) = T^i_j e^j(g)$ for all $g \in S$. Since S is admissible $S + iS$ is dense in $L^2[0,1]$ and $\mathcal{E}^j_S = \{e^j(g) : g \in S\}$ is total in $\mathfrak{H}^j$. The bounded operators S^i_j and T^i_j agree on $\mathcal{E}^j_S$ and therefore $S^i_j = T^i_j$. Therefore $S = T$ and T is the only cmx representation of $\hat{T}$. ∎

Similar considerations apply to the chaos decompositions of $\mathfrak{H}_{t]}$ and $\mathfrak{H}_{(t}$ and the corresponding exponential domains $\mathcal{E}_{t]}$ and $\mathcal{E}_{(t}$.

3 Basic Processes

In quantum stochastic analysis the basic integrators have particularly simple cmx representations: diagonal, subdiagonal and superdiagonal. This and Lemma 2.9 make feasible a chaos matrix based theory of quantum stochastic integration compatible with the Hudson–Parthasarathy theory.

If $J \in \mathcal{B}(L^2(\mathbb{R}_+))$ define $\Lambda(J)^j_j$ in $\mathcal{B}(\mathfrak{H}^j)$ by putting $\Lambda(J)^0_0 = 0$ and

$$(\Lambda(J)^j_j \psi)(x_1, \ldots, x_j) = \sum_{\nu=1}^{j} (J_\nu \psi)(x_1, \ldots, x_j),$$

whenever $\psi \in \mathfrak{H}^j$, $j = 1, 2, \ldots$. In this formula J_ν acts on ψ by fixing all variables other than x_ν and then applying J. The theory of Lebesgue integration shows that J_ν is in $\mathcal{B}(\mathfrak{H})$ with $\|J_\nu\| = \|J\|$ for $j = 1, 2, \ldots$ and $\nu = 1, \ldots, j$. Therefore $\Lambda(J)^j_j$ is well defined with $\|\Lambda(J)^j_j\| = j\,\|J\|$.

The second quantisation of J [5, §5.2.1] is represented by the chaos matrix

$$\Lambda(J) = \mathrm{diag}[0, \Lambda(J)^1_1, \Lambda(J)^2_2, \Lambda(J)^3_3, \ldots].$$

If J is self-adjoint then $\Lambda(J)^j_j$ is self-adjoint for all $j \in \mathbb{N}$ and $\Lambda(J)^* = \Lambda(J)$.

If $f \in L^2[0,1]$ define the adjoint pair, $a(f)^j_{j+1}$ in $\mathcal{B}(\mathfrak{H}^{j+1}, \mathfrak{H}^j)$ and $a^\dagger(f)^{j+1}_j$ in $\mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^{j+1})$, by putting

$$\begin{aligned}
(a(f)^j_{j+1}\psi)(x_1, \ldots, x_j) &= (j+1)^{1/2} \int_0^1 \overline{f(s)}\psi(x_1, \ldots, x_j, s)\, ds, \\
(a^\dagger(f)^{j+1}_j \varphi)(x_1, \ldots, x_{j+1}) &= (j+1)^{-1/2} \sum_{\nu=1}^{j+1} f(x_\nu)\varphi(x_1, \ldots, \hat{x}_\nu, \ldots, x_{j+1}),
\end{aligned}$$

whenever, $j = 1, 2, \ldots$, $\psi \in \mathfrak{H}^{j+1}$ and $\varphi \in \mathfrak{H}^j$. The 'hat' over x_ν in the above formula suppresses x_ν. Then

$$\left\|a(f)^j_{j+1}\right\| = (j+1)^{1/2}\,\|f\|\,, \qquad \left\|a^\dagger(f)^{j+1}_j\right\| = (j+1)^{1/2}\,\|f\|\,, \tag{18}$$

and the chaos matrices

$$\begin{bmatrix} 0 & a(f)^0_1 & 0 & 0 & \cdots \\ 0 & 0 & a(f)^1_2 & 0 & \cdots \\ 0 & 0 & 0 & a(f)^2_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ a^\dagger(f)^1_0 & 0 & 0 & 0 & \cdots \\ 0 & a^\dagger(f)^2_1 & 0 & 0 & \cdots \\ 0 & 0 & a^\dagger(f)^3_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

represent, respectively, the annihilation and creation operators $a(f)$ and $a^\dagger(f)$ on $\mathcal{D}(N^{1/2})$ [5, §5.2.1]. The same symbols $a(f)$ and $a^\dagger(f)$ denote the matrices above and the operators they represent.

If f is in $L^\infty[0,1]$ and $j \in \mathbb{N} \cup \{0\}$ then, for $\psi \in \mathfrak{H}^{j+1}$ and $\varphi \in L^2([0,1], \mathfrak{H}^j)$, the formulae

$$\begin{aligned} \nabla(f)^j_{j+1}(\psi)(s)(x_1,\ldots,x_j) &= (j+1)^{\frac{1}{2}}\psi(x_1,\ldots,x_j,s)\overline{f(s)}, \\ \mathcal{S}(f)^{j+1}_j(\varphi)(x_1,\ldots,x_{j+1}) &= (j+1)^{-\frac{1}{2}}\sum_{k=1}^{j+1} f(x_k)\varphi(x_k)(x_1,\ldots,\hat{x}_k,\ldots,x_{j+1}), \end{aligned}$$

define an adjoint pair of bounded linear transformations

$$\nabla(f)^j_{j+1} : \mathfrak{H}^{j+1} \to L^2([0,1],\mathfrak{H}^j), \qquad \mathcal{S}(f)^{j+1}_j : L^2([0,1],\mathfrak{H}^j) \to \mathfrak{H}^{j+1}.$$

A simple calculation shows that

$$\left\|\nabla(f)^j_{j+1}\right\| = (j+1)^{1/2}\left\|f\right\|_\infty, \qquad \left\|\mathcal{S}(f)^{j+1}_j\right\| = (j+1)^{1/2}\left\|f\right\|_\infty. \quad (19)$$

The chaos matrices

$$\begin{bmatrix} 0 & \nabla(f)^0_1 & 0 & 0 & \cdots \\ 0 & 0 & \nabla(f)^1_2 & 0 & \cdots \\ 0 & 0 & 0 & \nabla(f)^2_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \mathcal{S}(f)^1_0 & 0 & 0 & 0 & \cdots \\ 0 & \mathcal{S}(f)^2_1 & 0 & 0 & \cdots \\ 0 & 0 & \mathcal{S}(f)^3_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (20)$$

represent $\nabla(f)$, the *gradient operator*, and $\mathcal{S}(f)$, the *Hitsuda–Skorohod* operator, on $\mathcal{D}(N^{1/2})$ [14, §1.]. We dispense with the " ˆ " denoting operator and let $\nabla(f)$ and $\mathcal{S}(f)$ stand for the matrices and for the operators they represent. Similarly we write N for the number operator.

It follows from (19) that $(N+1)^{-1/2}\nabla(f)$ and $\mathcal{S}(f)(N+1)^{-1/2}$ are bounded in norm by $\|f\|_\infty$.

If $t \in [0,1]$ then

$$\nabla_t \stackrel{\text{def}}{=} \nabla(\chi_{[0,t]}), \qquad \mathcal{S}_t \stackrel{\text{def}}{=} \mathcal{S}(\chi_{[0,t]}$$

We will sometimes write $\nabla(f)(s)\varphi$ for $(\nabla(f)\varphi)(s)$ and $\nabla_t(s)\varphi$ for $\nabla_t(\varphi)(s)$.

The operators $\nabla(f)$ and $\mathcal{S}(f)$ are formally adjoint [14]. This is reflected in (21) below. The identity (23) is the key to the quantum Ito formula.

Proposition 3.1 *Let $f, g \in L^2[0,1]$. If $\psi, \theta \in L^2([0,1], \mathfrak{H}^j)$, $\phi \in L^2([0,1], \mathfrak{H}^{j+1})$ and $\varphi \in \mathfrak{H}^{j+1}$ then*

$$\langle \mathcal{S}(f)\psi, \varphi\rangle = \int_0^1 \langle \psi(s), \nabla(f)(s)\varphi\rangle \, ds = \langle \psi, \nabla(f)\varphi\rangle ; \quad (21)$$

$$\int_0^1 \langle \mathcal{S}(f)\psi, \phi(u)\rangle \, du = \int_0^1 \int_0^1 \langle \psi(s), \nabla(f)(s)\phi(u)\rangle \, du\, ds; \quad (22)$$

$$\langle \mathcal{S}(f)\psi, \mathcal{S}(g)\theta\rangle = \int_0^1 \int_0^1 \left\langle \nabla(\overline{f})(u)\psi(s), \nabla(\overline{g})(s)\theta(u)\right\rangle \, du\, ds + \int_0^1 \langle f(s)\psi(s), g(s)\theta(s)\rangle \, . \quad (23)$$

Proof. For each $s \in [0,1]$ the functions $\psi(s)$ and φ are completely symmetric. Therefore $\langle \mathcal{S}(f)\psi, \varphi\rangle$ equals

$$\begin{aligned}
&(j+1)^{-\frac{1}{2}} \sum_{k=1}^{j+1} \int_0^1 f(x_k)\psi(x_k)(x_1, \ldots, \hat{x}_k, \ldots, x_{j+1})\overline{\varphi(x_1, \ldots, x_{j+1})}\, dx \\
&= (j+1)^{\frac{1}{2}} \int_0^1 \int_{[0,1]^j} \psi(s)(y_1, \ldots, y_j)\overline{\varphi(y_1, \ldots, y_j, s)\overline{f(s)}}\, dy\, ds \\
&= \int_0^1 \langle \psi(s), \nabla(f)(s)\varphi\rangle \, ds,
\end{aligned}$$

where $dx = dx_1 \ldots dx_{j+1}$ and $dy = dy_1 \ldots dy_j$, proving (21). This leads immediately to (22). To prove (23) note that $(j+1)\langle \mathcal{S}(f)\psi, \mathcal{S}(g)\theta\rangle$ is equal to

$$\begin{aligned}
&\sum_{k,l=1}^{j+1} \int_{[0,1]^{j+1}} f(x_k)\psi(x_k)(x_1, \ldots, \hat{x}_k, \ldots, x_{j+1})\overline{g(x_l)\theta(x_l)(x_1, \ldots, \hat{x}_l, \ldots, x_{j+1})}\, dx \\
&= \sum_{k\neq l=1}^{j+1} \int_{[0,1]^{j+1}} f(x_k)\psi(x_k)(x_1, \ldots, \hat{x}_k, \ldots, x_{j+1})\overline{g(x_l)\theta(x_l)(x_1, \ldots, \hat{x}_l, \ldots, x_{j+1})}\, dx \\
&\quad + \sum_{k=1}^{j+1} \int_{[0,1]^{j+1}} f(x_k)\psi(x_k)(x_1, \ldots, \hat{x}_k, \ldots, x_{j+1})\overline{g(x_k)\theta(x_k)(x_1, \ldots, \hat{x}_k, \ldots, x_{j+1})}\, dx \\
&= (j+1)\int_0^1\int_0^1 \left\langle \nabla(\overline{f})(u)\psi(s), \nabla(\overline{g})(s)\theta(u)\right\rangle \, du\, ds + (j+1)\int_0^1 \langle f(s)\psi(s), g(s)\theta(s)\rangle \, . \blacksquare
\end{aligned}$$

The proof of (22) yields the following corollary.

Corollary 3.2 *If $f \in L^\infty[0,1]$ and $j \in \mathbb{N} \cup \{0\}$ then*

$$\left(\nabla(f)^j_{j+1}\right)^* = \mathcal{S}(f)^{j+1}_j.$$

If Ψ is one of the basic matrices $\Lambda(f)$, $a(f)$, $a^\dagger(f)$, $\nabla(f)$ and $\mathcal{S}(f)$ then Ψ is tridiagonal. Therefore, the product chaos matrices ΨT and $T\Psi$ exist for each chaos matrix T. For example the formulae

$$\begin{aligned} ((\nabla(f)T)^{i-1}_j \psi)(s)(x_1, \ldots, x_{i-1}) &= (\nabla(f)^{i-1}_i(s) T^i_j \psi)(x_1, \ldots, x_{i-1}); \\ ((T\nabla(f))^i_j \psi)(s)(x_1, \ldots, x_i) &= (T^i_{j-1}(\nabla(f)^{j-1}_j \psi)(s))(x_1, \ldots, x_i), \end{aligned}$$

whenever $\psi \in \mathfrak{H}^j$ define the products (composition) of the chaos matrices $\nabla(f)T$ and $T\nabla(f)$. The other products are similarly defined.

4 Adaptedness

Let $\hat{T} = \{\hat{T}_t : t \in [0,1]\}$ be a process of operators in $\mathcal{L}(\mathcal{H})$ which is adapted in the sense of Hudson and Parthasarathy [11, Definition 3.1]. Then $\hat{T}_t$ is the ampliation of an operator $\hat{T}_{t]}$ in $\mathcal{L}(\mathfrak{H}_{t]})$ for all t. A possible definition of adaptedness for a cmx process T would be that the corresponding operator-valued process $\tilde{T}$ be adapted. Our proof of the quantum Ito formula for cmx semimartingales requires a more intrinsic definition of adaptedness and, in particular, a description of ampliation in terms of chaos matrices. This intrinsic definition is shown to be compatible with the Hudson-Parthasarathy definition.

For each $t \in [0,1]$ there is a unique unitary $\mathfrak{U} = \mathfrak{U}_t : \mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t} \hookrightarrow\!\!\!\!\rightarrow \mathfrak{H}$ satisfying

$$\mathfrak{U}(e(g_{t]}) \otimes e(g_{(t})) = e(g_{t]} + g_{(t}) \qquad \forall \quad g \in L^2[0,1] \tag{24}$$

where $g_{t]} = \chi_{[0,t]} g$ and $g_{(t} = \chi_{(t,1]} g$. The map $\mathfrak{U}_t$ is natural and defined at the algebraic level:

$$\mathcal{E} = \mathfrak{U}_t(\mathcal{E}_{t]} \underline{\otimes} \mathcal{E}_{(t})$$

where $\mathcal{E}$, $\mathcal{E}_{t]}$ and $\mathcal{E}_{(t}$ are the exponential domains of $\mathfrak{H}$, $\mathfrak{H}_{t]}$ and $\mathfrak{H}_{(t}$ respectively and '$\underline{\otimes}$' denotes *algebraic* tensor product.

For the rest of this section fix $t \in [0,1]$ and put $\mathfrak{U} = \mathfrak{U}_t$. Let $\hat{T}_{t]}$ be a linear transformation in $\mathfrak{H}_{t]}$ with domain $\mathcal{D}(\hat{T}_{t]})$. *The ampliation* of $\hat{T}_{t]}$ to $\mathfrak{U}[\mathcal{D}(T_{t]})\underline{\otimes}\mathfrak{H}_{(t}]$ is the linear transformation $\hat{T}_t \stackrel{\text{def}}{=\!=} \mathfrak{U} \circ (\hat{T}_{t]}\underline{\otimes}\hat{I}_{(t)}) \circ \mathfrak{U}^{-1}$. Thus for $\alpha \in \mathcal{D}(\hat{T}_{t]})$ and $\beta \in \mathfrak{H}_{(t}$

$$\hat{T}\mathfrak{U}[\alpha \otimes \beta] = \mathfrak{U}[(\hat{T}_{t]}\alpha) \otimes \beta].$$

Any extension of $\hat{T}_t$ is said to be *an ampliation* of $\hat{T}_{t]}$.

If $\hat{T}_{t]}$ is bounded with cmx representation $T_{t]}$ there is a unique bounded chaos matrix T_t such that $\tilde{T}_t$ is an ampliation of $\hat{T}_{t]}$ to $\mathfrak{H}$. Therefore the entries of T_t may be represented in terms of the entries of $T_{t]}$. To find this representation we use the deconstruction of $\mathfrak{U}$ described in Proposition 2.8.

It follows from Proposition 2.8 that if $\alpha \in \mathfrak{H}_{t]}^{j-r}$ and $\beta \in \mathfrak{H}_{(t}^{r}$ then

$$\mathfrak{U}(\alpha \otimes \beta) = \begin{pmatrix} j \\ r \end{pmatrix}^{\frac{1}{2}} \alpha \otimes_{\text{sym}} \beta. \tag{25}$$

Moreover $\mathfrak{H}^j$ is the closed linear span of all vectors of the form

$$\psi^j \quad = \quad \alpha^j \otimes_{\text{sym}} \beta^0 \oplus \cdots \oplus \alpha^{j-r} \otimes_{\text{sym}} \beta^r \oplus \cdots \oplus \alpha^0 \otimes_{\text{sym}} \beta^j, \tag{26}$$

where $\alpha^{j-r} \in \mathfrak{H}_{t]}^{j-r}$ and $\beta^r \in \mathfrak{H}_{(t}^{r}$.

If $\alpha, \gamma \in \mathfrak{H}_{t]}^{i-r}$ and $\beta, \delta \in \mathfrak{H}_{(t}^{r}$ then

$$\|\alpha \otimes \beta\| = \|\mathfrak{U}(\alpha \otimes \beta)\| = \begin{pmatrix} i \\ r \end{pmatrix}^{\frac{1}{2}} \|\alpha \otimes_{\text{sym}} \beta\|,$$

and, by polarisation,

$$\begin{pmatrix} i \\ r \end{pmatrix} \langle \alpha \otimes_{\text{sym}} \beta, \gamma \otimes_{\text{sym}} \delta \rangle = \langle \alpha, \gamma \rangle \langle \beta, \delta \rangle .$$

This implies that if $S \in \mathcal{B}(\mathfrak{H}_{t]}^{j-r}, \mathfrak{H}_{t]}^{i-r})$ then

$$\begin{pmatrix} i \\ r \end{pmatrix} \langle (S\alpha) \otimes_{\text{sym}} \beta, \gamma \otimes_{\text{sym}} \delta \rangle \quad = \quad \begin{pmatrix} j \\ r \end{pmatrix} \left\langle \alpha \otimes_{\text{sym}} \beta, (S^*\gamma) \otimes_{\text{sym}} \delta \right\rangle, \tag{27}$$

whenever $\alpha \in \mathfrak{H}_{t]}^{j-r}$, $\gamma \in \mathfrak{H}_{t]}^{i-r}$ and $\beta, \delta \in \mathfrak{H}_{(t}^{r}$.

Define $S \otimes_{\text{sym}} I^r_{(t}$ in $\mathcal{B}(\mathfrak{H}^{j-r}_{t]} \otimes_{\text{sym}} \mathfrak{H}^r_{(t}, \mathfrak{H}^{i-r}_{t]} \otimes_{\text{sym}} \mathfrak{H}^r_{(t})$, where $I^r_{(t}$ is the identity in $\mathfrak{H}^r_{(t}$, by putting

$$(S \otimes_{\text{sym}} I^r_{(t})(\alpha \otimes_{\text{sym}} \beta) = (S\alpha) \otimes_{\text{sym}} \beta$$

whenever $\alpha \in \mathfrak{H}^{j-r}_{t]}$ and $\beta \in \mathfrak{H}^r_{(t}$. It follows from (25) that

$$S \otimes_{\text{sym}} I^r_{(t} \quad = \quad \begin{pmatrix} i \\ r \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} j \\ r \end{pmatrix}^{\frac{1}{2}} \mathfrak{U}(S \otimes I^r_{(t})\mathfrak{U}^{-1}. \tag{28}$$

We obtain following lemma directly from (27).

Lemma 4.1 *If $S \in \mathcal{B}(\mathfrak{H}^{j-r}_{t]}, \mathfrak{H}^{i-r}_{t]})$ then*

$$\begin{pmatrix} i \\ r \end{pmatrix} (S \otimes_{\text{sym}} I^r_{(t})^* = \begin{pmatrix} j \\ r \end{pmatrix} (S^* \otimes_{\text{sym}} I^r_{(t}). \tag{29}$$

Theorem 4.2 *Let $T_{t]}$ be a chaos matrix for $\mathfrak{H}_{t]}$ and let T_t be the chaos matrix whose entries are defined by the formula*

$$T_t{}^i_j = \bigoplus_{r=0}^{i\wedge j} \begin{pmatrix} i \\ r \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} j \\ r \end{pmatrix}^{-\frac{1}{2}} (T_{t]}{}^{i-r}_{j-r} \otimes_{\text{sym}} I^r_{(t}) = \bigoplus_{r=0}^{i\wedge j} \mathfrak{U} \circ (T_{t]}{}^{i-r}_{j-r} \otimes I^r_{(t}) \circ \mathfrak{U}^{-1}. \tag{30}$$

Then

(i) $\mathfrak{U}[\mathfrak{D}(T_{t]})\underline{\otimes}\mathfrak{H}_{(t}] \subset \mathfrak{D}(T_t)$*;*

(ii) $\tilde{T}_t$ *is an ampliation of* $\tilde{T}_{t]}$ *on* $\mathfrak{U}[\mathfrak{D}(T_{t]})\underline{\otimes}\mathfrak{H}_{(t}]$*.*

Proof. If $\alpha \in \mathfrak{D}(T_{t]})$ and $\beta \in \mathfrak{H}_{(t}$ then $\eta^i = (\tilde{T}_{t]}\alpha)^i = \sum_{j=0}^{\infty} T_{t]}{}^i_j \alpha^j$ where the sum is convergent in $\mathfrak{H}^i$. Moreover $\eta = (\eta^0, \eta^1, \ldots, \eta^i, \ldots)^t \in \mathfrak{H}$. Therefore

$$\begin{aligned} \Big((\tilde{T}_{t]}\alpha) \otimes \beta\Big)^i &= \bigoplus_{r=0}^{i} (T_{t]}\alpha)^{i-r} \otimes \beta^r \\ &= \bigoplus_{r=0}^{i} \sum_{j=0}^{\infty} (T_{t]}{}^{i-r}_j \alpha^j) \otimes \beta^r \end{aligned}$$

$$
\begin{aligned}
&= \bigoplus_{r=0}^{i} \sum_{j=r}^{\infty} (T_{t]}{}^{i-r}_{j-r}\alpha^{j-r}) \otimes \beta^r \\
&= \sum_{j=0}^{\infty} \bigoplus_{r=0}^{i\wedge j} (T_{t]}{}^{i-r}_{j-r}\alpha^{j-r}) \otimes \beta^r \\
&= \sum_{j=0}^{\infty} \bigoplus_{r=0}^{i\wedge j} (T_{t]}{}^{i-r}_{j-r} \otimes I^r_{(t})(\alpha \otimes \beta).
\end{aligned}
$$

These sums are all convergent in $\mathfrak{H}_{t]} \otimes \mathfrak{H}_{(t}$. Therefore

$$
\begin{aligned}
\Big(\mathfrak{U}[(\tilde{T}_{t]}\alpha) \otimes \beta]\Big)^i &= \mathfrak{U}([(\tilde{T}_{t]}\alpha) \otimes \beta]^i) \\
&= \sum_{j=0}^{\infty} \bigoplus_{r=0}^{i\wedge j} \mathfrak{U}\left[(T_{t]}{}^{i-r}_{j-r} \otimes I^r_{(t})(\alpha \otimes \beta)\right] \\
&= \sum_{j=0}^{\infty} \bigoplus_{r=0}^{i\wedge j} \begin{pmatrix} i \\ r \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} j \\ r \end{pmatrix}^{-\frac{1}{2}} (T_{t]}{}^{i-r}_{j-r} \otimes_{\text{sym}} I^r_{(t})\mathfrak{U}[(\alpha \otimes \beta)] \qquad (31) \\
&= \sum_{j=0}^{\infty} T_t{}^i_j (\mathfrak{U}[\alpha \otimes \beta])^j. \qquad (32)
\end{aligned}
$$

Equation (31) follows from (28) and the sums are all convergent in $\mathfrak{H}^i$. Now

$$
\begin{aligned}
\bigoplus_{i=0}^{\infty} \Big((\tilde{T}_{t]}\alpha) \otimes \beta\Big)^i &= \bigoplus_{i=0}^{\infty} \bigoplus_{r=0}^{i} (\tilde{T}_{t]}\alpha)^{i-r} \otimes \beta^r \\
&= \bigoplus_{r=0}^{\infty} \bigoplus_{i=r}^{\infty} (\tilde{T}_{t]}\alpha)^{i-r} \otimes \beta^r \\
&= \bigoplus_{r=0}^{\infty} \bigoplus_{i=0}^{\infty} (\tilde{T}_{t]}\alpha)^{i} \otimes \beta^r
\end{aligned}
$$

which is convergent in $\mathfrak{H}_{t]} \underline{\otimes} \mathfrak{H}_{(t}$ to $(\tilde{T}_{t]}\alpha) \otimes \beta$. Therefore

$$
\bigoplus_{i=0}^{\infty} \mathfrak{U}\left[\Big((\tilde{T}_{t]}\alpha) \otimes \beta\Big)^i\right]
$$

is convergent in $\mathfrak{H}$. It follows from (32) that $\mathfrak{U}[\alpha \otimes \beta]$ belongs to $\mathfrak{D}(T_t)$, proving (i), and also that

$$\mathfrak{U}\left[(\tilde{T}_{t]}\alpha) \otimes \beta\right] = \tilde{T}_t \mathfrak{U}[\alpha \otimes \beta].$$

Therefore $\tilde{T}_t$ agrees with the ampliation of $T_{t]}$ on $\mathfrak{D}(T_{t]})\underline{\otimes}\mathfrak{H}_{(t}$, proving (ii). ∎

We define the *ampliation* of $T_{t]}$ to be chaos matrix T_t defined by formula (30). In this case we write

$$T_t = T_{t]} \otimes_{\text{sym}} I_{(t}.$$

Note that T_t is $(2k+1)$-diagonal whenever $T_{t]}$ is $(2k+1)$-diagonal.

A cmx process T is *adapted* if for each $t \in [0,1]$ there exists a chaos matrix $T_{t]}$ such that T_t is the ampliation of $T_{t]}$ for each $t \in [0,1]$.

It was shown in Theorem 2.2 (ii) that the map $T \mapsto \tilde{T}$ is an isometric isomorphism from $\mathcal{B}_{\text{cmx}}(\mathfrak{H}_{t]})$ onto $\mathcal{B}(\mathfrak{H}_{t]})$ for each $t \in [0,1]$. This leads to the following corollaries.

Corollary 4.3 *Let T_t be a bounded chaos matrix. Then T_t is the ampliation of $T_{t]} \in \mathcal{B}_{\text{cmx}}(\mathfrak{H}_{t]})$ if and only if $\tilde{T}_t$ is the ampliation of $\tilde{T}_{t]}$.*

Corollary 4.4 *$T = \{T_t : t \in [0,1]\}$ is a regular adapted cmx process if and only $\tilde{T} = \{\tilde{T}_t : t \in [0,1]\}$ is a regular adapted operator process.*

Using the gradient operator $\nabla \stackrel{\text{def}}{=} \nabla(\chi_{[0,1]})$ we give a characterisation of ampliation which is significantly easier to use in computation than (30).

In what follows $\nabla(s)(\psi)$ denotes $(\nabla(\psi))(s)$ whenever $\psi \in \mathcal{D}(N^{1/2})$.

Let $\mathcal{M}^j_{j-r}$, $j \geq r$ be the set of strictly monotone functions $\pi : \{1, 2, \ldots, j-r\} \to \{1, 2, \ldots, j\}$ and if $\pi \in \mathcal{M}^j_{j-r}$ let $\check{\pi}$ be the unique function in C^j_r whose image is disjoint from that of π. Note that $|\mathcal{M}^j_r| = |\mathcal{M}^j_{j-r}| = \begin{pmatrix} j \\ r \end{pmatrix}$.

Lemma 4.5 *If* $\alpha \in \mathfrak{H}^{j-r}_{t]}$, $\beta \in \mathfrak{H}^r_{(t}$ *and* $s_1, \ldots, s_r \in (t, 1]$ *then*

$$\nabla(s_1)(\alpha \otimes_{\text{sym}} \beta) = \frac{r^{\frac{1}{2}}}{j^{\frac{1}{2}}} \alpha \otimes_{\text{sym}} \nabla(s_1)\beta; \tag{33}$$

$$\nabla(s_1) \cdots \nabla(s_r)(\alpha \otimes_{\text{sym}} \beta) = \binom{j}{r}^{-\frac{1}{2}} (\alpha \otimes_{\text{sym}} 1)\beta(s_1, \ldots, s_r), \tag{34}$$

where 1 *is the unit in* $\mathfrak{H}^0_{(t} = \mathfrak{H}^0 = \mathbb{C}$.

Proof.

$$\begin{aligned}
\nabla(s_1)(\alpha \otimes_{\text{sym}} \beta)(x_1, \ldots, x_{j-1}) & \\
&= j^{\frac{1}{2}} \binom{j}{r}^{-1} \sum_{\substack{\pi \in \mathcal{M}^j_{j-r} \\ \check{\pi}_r = j}} \alpha(x_{\pi_1} \cdots x_{\pi_{j-r}})\beta(x_{\check{\pi}_1} \cdots x_{\check{\pi}_{r-1}}, s_1) \\
&= j^{\frac{1}{2}} \binom{j}{r}^{-1} \sum_{\pi \in \mathcal{M}^{j-1}_{j-r}} \alpha(x_{\pi_1} \cdots x_{\pi_{j-r}})\beta(x_{\check{\pi}_1} \cdots x_{\check{\pi}_{r-1}}, s_1) \\
&= \frac{j^{\frac{1}{2}}}{r^{\frac{1}{2}}} \binom{j}{r}^{-1} \sum_{\pi \in \mathcal{M}^{j-1}_{j-r}} \alpha(x_{\pi_1} \cdots x_{\pi_{j-r}})(\nabla(s_1)\beta)(x_{\check{\pi}_1} \cdots x_{\check{\pi}_{r-1}}) \\
&= \frac{j^{\frac{1}{2}}}{r^{\frac{1}{2}}} \binom{j}{r}^{-1} \binom{j-1}{r-1} (\alpha \otimes_{\text{sym}} \nabla(s_1)\beta)(x_1, \ldots, x_{j-1}) \\
&= \frac{r^{\frac{1}{2}}}{j^{\frac{1}{2}}} (\alpha \otimes_{\text{sym}} \nabla(s_1)\beta)(x_1, \ldots, x_{j-1}),
\end{aligned}$$

proving (33). Repeated applications of (33) give (34). ∎

Theorem 4.6 *A chaos matrix* T_t *for* $\mathfrak{H}$ *is the ampliation of a chaos matrix* $T_{t]}$ *for* $\mathfrak{H}_{t]}$ *if and only if, for each* $s \in (t, 1]$,

$$\text{(i) } T_t \nabla(s) = \nabla(s) T_t \quad \textit{and} \quad \text{(ii) } T_t^* \nabla(s) = \nabla(s) T_t^*. \tag{35}$$

In this case, if $i, j \in \mathbb{N}$, *then*

$$T_{t]}{}^i_j = \chi_{[0,t]^j} T_t{}^i_j \chi_{[0,t]^i}. \tag{36}$$

Proof. If T_t is the ampliation of $T_{t]}$ and $\alpha^{j-r} \in \mathfrak{H}_{t]}^{j-r}$ and $\beta^r \in \mathfrak{H}_{(t}^r$ then

$$\begin{aligned}
\nabla(s) T_t{}^i_j(\alpha^{j-r} \otimes_{\text{sym}} \beta^r) &= \binom{i}{r}^{\frac{1}{2}} \binom{j}{r}^{-\frac{1}{2}} \nabla(s)(T_{t]}{}^{i-r}_{j-r} \alpha^{j-r} \otimes_{\text{sym}} \beta^r) \\
&= \left(\frac{r}{i}\right)^{\frac{1}{2}} \binom{i}{r}^{\frac{1}{2}} \binom{j}{r}^{-\frac{1}{2}} (T_{t]}{}^{i-r}_{j-r} \alpha^{j-r} \otimes_{\text{sym}} \nabla(s)\beta^r) \\
&= \left(\frac{r}{j}\right)^{\frac{1}{2}} \binom{i-1}{r-1}^{\frac{1}{2}} \binom{j-1}{r-1}^{-\frac{1}{2}} (T_{t]}{}^{i-r}_{j-r} \alpha^{j-r} \otimes_{\text{sym}} \nabla(s)\beta^r) \\
&= \left(\frac{r}{j}\right)^{\frac{1}{2}} T_t{}^{i-1}_{j-1}(\alpha^{j-r} \otimes_{\text{sym}} \nabla(s)\beta^r) \\
&= T_t{}^{i-1}_{j-1} \nabla(s)(\alpha^{j-r} \otimes_{\text{sym}} \beta^r).
\end{aligned}$$

It follows by linearity that $T_t{}^{i-1}_{j-1}\nabla(s)\psi^j = \nabla(s)T_t{}^i_j\psi^j$ almost everywhere for all ψ^j of the form (26). Since such ψ^j are total in $\mathfrak{H}^j$ the identity (35)(i) is valid.

It follows from (28) and definition (30) that T_t^* is the ampliation of $T_{t]}^*$. Therefore the identity (35)(ii) is also valid.

Suppose conversely that T_t satisfies the identities (35) (i) and (ii) and define $T_{t]}$ by (36). If $\alpha^{j-r} \in \mathfrak{H}_{t]}^{j-r}$ and $\beta^r \in \mathfrak{H}_{(t}^r$ and $i \geq j$ then

$$\begin{aligned}
\nabla(s_1)\cdots\nabla(s_r) T_t{}^i_j(\alpha \otimes_{\text{sym}} \beta) &= T_t{}^{i-r}_{j-r} \nabla(s_1)\cdots\nabla(s_r)(\alpha \otimes_{\text{sym}} \beta) \\
&= \binom{j}{r}^{-\frac{1}{2}} T_t{}^{i-r}_{j-r}(\alpha \otimes_{\text{sym}} 1)\beta(s_1,\ldots,s_r), \\
&= \binom{j}{r}^{-\frac{1}{2}} (T_{t]}{}^{i-r}_{j-r}\alpha \otimes_{\text{sym}} 1)\beta(s_1,\ldots,s_r), \\
&= \binom{i}{r}^{\frac{1}{2}} \binom{j}{r}^{-\frac{1}{2}} \nabla(s_1)\cdots\nabla(s_r)(T_{t]}{}^{i-r}_{j-r}\alpha \otimes_{\text{sym}} \beta).
\end{aligned}$$

The third equality is valid since $\nabla(s)T_t{}^{i-r}_{j-r}(\alpha \otimes_{\text{sym}} 1) = 0$ whenever $s \in (t, 1]$. Therefore

$$T_t{}^i_j = \sum_{r=0}^{j} \binom{i}{r}^{\frac{1}{2}} \binom{j}{r}^{-\frac{1}{2}} (T_{t]}{}^{i-r}_{j-r} \otimes_{\text{sym}} I_{(t)}^r) \tag{37}$$

whenever $i \geq j$. Now put $S_t = {T_t}^*$, apply (37) to S_t, take adjoints and use (29) to give

$$\begin{aligned} {T_t}^{j}_{i} = (S^i_j)^* &= \sum_{r=0}^{i} \binom{i}{r}^{\frac{1}{2}} \binom{j}{r}^{-\frac{1}{2}} (S_{t]}{}^{i-r}_{j-r} \otimes_{\mathrm{sym}} I^r_{(t)})^* \\ &= \sum_{r=0}^{i} \binom{j}{r}^{\frac{1}{2}} \binom{i}{r}^{-\frac{1}{2}} ((S_{t]}{}^{i-r}_{j-r})^* \otimes_{\mathrm{sym}} I^r_{(t)}) \\ &= \sum_{r=0}^{i} \binom{j}{r}^{\frac{1}{2}} \binom{i}{r}^{-\frac{1}{2}} (S^*_{t]}{}^{j-r}_{i-r} \otimes_{\mathrm{sym}} I^r_{(t)}) \\ &= \sum_{r=0}^{i} \binom{j}{r}^{\frac{1}{2}} \binom{i}{r}^{-\frac{1}{2}} (T_{t]}{}^{j-r}_{i-r} \otimes_{\mathrm{sym}} I^r_{(t)}). \end{aligned}$$

Thus Equation (37) is valid for all i, j and T_t is the ampliation of $T_{t]}$ defined by Formula (30). ∎

The characterisation (35) does not transfer directly to operators $\tilde{T}$ since there is no guarantee that the product operators $\nabla(s)\tilde{T}_t$ exist on the domain of $\nabla(s)$ even when $\tilde{T}_t$ is bounded. At the formal level it is closely related to the extended definition of adaptedness given by Attal [2, §I.1.2].

It is also related to [**?**, Definition 4.3.] which appears to be an integrated version of (35)

Corollary 4.7 *If S and T are adapted cmx processes then*

(i) T^* *is adapted;*

(ii) *the product process ST is adapted whenever it exists.*

Proof. (i) It is immediate from (35) that T is adapted if and only if T^* is adapted.

(ii) The product process ST exists if and only if $(ST)^*$ exists. Since S and T are adapted

$$S_t T_t \nabla(s) = S_t \nabla(s) T_t = \nabla(s) S_t T_t,$$

and

$$T^*_t S^*_t \nabla(s) = T^*_t \nabla(s) S^*_t = \nabla(s) T^*_t S^*_t,$$

so that $(ST)_t \nabla(s) = \nabla(s)(ST)_t$ and $(ST)^*_t \nabla(s) = \nabla(s)(ST)^*_t$ whenever $s > t$. Therefore $(ST)_t$ satisfies (35) and ST is an adapted process. ∎

5 Processes defined by Kernels

A large class of adapted processes in $\mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})$ may be defined using L^2 kernels. Such kernel processes may be used to provide non-trivial examples of quantum semimartingales which satisfy the quantum Duhamel and quantum Ito formulae. They should not be confused with the "integral kernel operators" studied by Obata[§4.3]OB1 For $i, j \in \mathbb{N} \cup \{0\}$ let

$$k_t{}^i_j(x; y) \equiv k^i_j(x; y; t), \quad (x; y; t) \in [0,1]^i \times [0,1]^j \times [0,1],$$

where $x = (x_1, \ldots, x_i)$, $y = (y_1, \ldots, y_j)$, be a Lebesgue-measurable kernels completely symmetric in both $x_1, \ldots, x_i$ and $y_1, \ldots, y_j$. Suppose that $k_t{}^i_j$ is bounded in the L^2-norm, for each $t \in [0,1]$:

$$\left\|k_t{}^i_j\right\| = \left(\int_{[0,1]^i \times [0,1]^j} |k^i_j(x; y; t)|^2 \, dx \, dy\right)^{1/2} < \infty,$$

and let α_t be the scalar matrix with entries $\alpha_t{}^i_j = \|k_t{}^i_j\|$. If $t \in [0,1]$ let $k_{t]}{}^i_j$ be the restriction of k^i_j to $[0,t]^i \times [0,t]^j \times [0,t]$. The formula

$$(K_{t]}{}^i_j \theta)(x) \quad = \quad \int_{[0,1]^j} k_{t]}{}^i_j(x; y; t)\theta(y) \, dy, \quad \theta \in \mathfrak{H}^i_j, \tag{38}$$

defines $K_{t]}{}^i_j$ in $\mathcal{B}(\mathfrak{H}^j_{t]}, \mathfrak{H}^i_{t]})_{\mathrm{so}}$ with $\left\|K_{t]}{}^i_j\right\| \le \alpha_t{}^i_j$.

Theorem 5.1 *Let $K_{t]}$ be the chaos matrix for $\mathfrak{H}_{t]}$ with entries $K_{t]}{}^i_j$ defined by (**??**).*

(i) *Then $K = \{K_t : t \in [0,1]\}$ is an adapted cmx process where K_t is the ampliation of $K_{t]}$ defined by (30). bb*(ii) *If $\alpha^i_j \in L^p[0,1]$ for all i, j then $K \in \mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$.*

(iii) *If the matrix $[k_t{}^i_j]$ is $(2k+1)$-diagonal then so is K_t.*

The adapted cmx process K represents the adapted operator valued process $\tilde{K} = \{\tilde{K}_t : t \in [0,1]\}$. However, without extra conditions on K, the process $\tilde{K}$ need not be regular. Nor need the operators $\tilde{K}_t$ have a common dense domain.

Suppose α_t belongs to $\mathcal{B}_{\text{cmx}}(\ell^2)$ for each $t \in [0,1]$. If $\psi \in \mathfrak{H}_{t]}$ then $\nu(\psi) = (\|\psi^0\|, \|\psi^1\|, \|\psi^2\|, \ldots)^t$ belongs to ℓ^2 and the series $\sum_i \left(\sum_j \alpha^i_j(t)\,\|\psi^j\|\right)^2$ is convergent. For each $\psi \in \mathfrak{H}_{t]}$ define

$$
\begin{aligned}
(\tilde{K}_{t]}\psi)^i &= \sum_{j=0}^{\infty} K_{t]}{}^i_j \psi^j \\
\tilde{K}_{t]}\psi &= \bigoplus_{i=0}^{\infty} (K_{t]}\psi)^i
\end{aligned}
\tag{39}
$$

These series converge since $\nu(\psi) \in \ell^2$ and

$$
\sum_{i=0}^{\infty} \left\|(\tilde{K}_{t]}\psi)^i\right\|^2 \le \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\left\|K_{t]}{}^i_j\psi^j\right\|\right)^2 \le \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\alpha_t{}^i_j\,\|\psi^j\|\right)^2 \le \|\alpha_t\|^2\,\|\psi\|^2
$$

The operator $\tilde{K}_t$ is bounded with $\left\|\tilde{K}_t\right\| \le \|\alpha_t\|$. We immediately have the following proposition.

Proposition 5.2 *The formulae (39) define an adapted regular operator process $\tilde{K} = \{\tilde{K}_t : t \in [0,1]\}$ with $\tilde{K}_t = \tilde{K}_{t]} \otimes_{\text{sym}} I_{(t}$ such that $K_{t]}$ is a cmx representation of $\tilde{K}_{t]}$ and K_t is a cmx representation of $\tilde{K}_t$.*

If α belongs to the Bochner–Lebesgue space $L^p([0,1], \mathcal{B}_{\text{cmx}}(\ell^2))$ then $\tilde{K} \in L^p([0,1], \mathcal{B}(\mathfrak{H}))$.

Given a $(2k+1)$-diagonal matrix $\hat{k}^i_j$ of L^2-kernels with $\left\|k_t{}^i_j\right\| <\le C$ it is possible to construct a variety of $(2k+1)$-diagonal processes in $L^p([0,1], \mathcal{B}_{\text{cmx}}(\ell^2))$. Choose functions $t \mapsto c^i_j(t) = c^j_i(t)$ with $|c^i_j| < \alpha \in L^p[0,1]$ and put $k_t{}^i_j \stackrel{\text{def}}{=} c^i_j(t)\hat{k}_t{}^i_j$. Then the process K defined by (38) belongs to $L^p([0,1], \mathcal{B}_{\text{cmx}}(\ell^2))$.

If the matrices $\{\alpha_t : t \in [0,1]\}$ have common domain $\mathfrak{D} \subset \ell^2$ then $\tilde{K}$ is an operator process the operators $\{\tilde{K}_t : t \in [0,1]\}$ have common domain $\mathcal{D}$ such that $\{\psi \in \mathfrak{H} : \nu(\psi) \in \mathfrak{D}\} \subset \mathcal{D}$. If $\mathfrak{D}$ contains a dense subset of the positive cone in ℓ^2 then $\mathcal{D}$ is dense in $\mathfrak{H}$.

6 Quantum Stochastic Integrals

Lindsay [14, Definition 2.1, Proposition 2.4] extends Hudson and Parthasarathy's definition of quantum stochastic integrals [11] to include non-adapted integrands. Lindsay's integrals are defined in terms of the Hitsuda-Skorohod process $t \mapsto \mathcal{S}_t$ and the gradient process $t \mapsto \nabla_t$. The cmx representations of these two processes are given by (20). Their simple form facilitates a formal calculation of the cmx representations of Lindsay's quantum stochastic integrals. This leads us to definition of the cmx quantum stochastic integrals of cmx processes compatible with Lindsay's definition for operator processes.

The integrators are the quadruple $(\Lambda, A, A^\dagger, t)$ of basic processes:

the *gauge* process $\Lambda : t \to \Lambda(\chi_{[0,t]})$;

the *annihilation* process, $A : t \to a(\chi_{[0,t]})$;

the *creation* process, $A^\dagger : t \to a^\dagger(\chi_{[0,t]})$;

the *time* process $tI = t$.

In the definition of the gauge process $\chi_{[0,t]} \in \mathcal{B}(L^2[0,1])$ is the operator of multiplication by $\chi_{[0,t]}$;

A quadruple (E, F, G, H) of cmx processes is *integrable* if $E \in \mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, $F, G \in \mathcal{L}^2_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, and $H \in \mathcal{L}^1_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and *symmetric* if $E = E^*$, $G = F^*$ and $H = h^*$. The processes E, F, G, and H are integrands for $d\Lambda$, dA, $dA^\dagger$ and dt respectively. This is consistent with, but wider than, the definition, (57) of an integrable quadruple of operator processes introduced in [1].

For the sake of clarity and to prevent a 'debauch of indices' in the proofs we have kept to this old-fashioned notation rather than using Evans' notation [7], [16, V §2.1]. However Mr A Belton's extension of the Evans notation allows a very efficient proof of the local Ito Product formula, Lemma 9.1. In the higher dimensional case the benefits of the Evans-Belton notations are even greater.

Let (E,F,G,H) be an integrable quadruple of cmx processes. The formulae

$$M(E:d\Lambda) = \mathcal{S}E\nabla, \qquad M_t(E:d\Lambda)^{i+1}_{j+1} = \mathcal{S}_{t\,i}^{\,i+1}E^i_j\nabla_{t\,j+1}^{\,j}, \tag{40}$$

$$M(F:dA) = \int_0^1 F_s\nabla(s)\,ds, \qquad M_t(F:dA)^i_{j+1} = \int_0^1 F_{s\,j}^{\,i}\nabla_{t\,j+1}^{\,j}(s)\,ds, \tag{41}$$

$$M(dA^\dagger:G) = \mathcal{S}G, \qquad M_t(dA^\dagger:G)^{i+1}_j = \mathcal{S}_{t\,i}^{\,i+1}G^i_j, \tag{42}$$

$$M_t(H:\,dt) = \int_0^t H_s\,ds, \qquad M_t(H:\,dt)^i_j = \int_0^t H_{s\,j}^{\,i}\,ds, \tag{43}$$

$$M_t(E:d\Lambda)^0_j = M_t(E:d\Lambda)^i_0 = M_t(F:dA)^i_0 = M_t(dA^\dagger:G)^0_j = 0$$

where $i,j = 0,1,2,\ldots$, define cmx processes

$$M(E:d\Lambda), \quad M(F:dA), \qquad M(dA^\dagger:G), \qquad M(H,dt).$$

It may be helpful to think of these processes as cmx representations of the Hudson–Parthasarathy quantum stochastic integrals

$$\int_0^t \tilde{E}_s\,d\Lambda_s, \quad \int_0^t \tilde{F}_s\,dA_s, \quad \int_0^t dA_s^\dagger\tilde{G}_s, \quad \int_0^t \tilde{H}_s\,ds.$$

We examine the first three of these definitions in more detail. In (40) E^i_j denotes the mapping $E^i_j : L^2([0,1],\mathfrak{H}^j) \to L^2([0,1],\mathfrak{H}^i)$ defined by the formula

$$(E^i_j\varphi)(s) = E_{s\,j}^{\,i}\varphi(s), \qquad \varphi \in L^2([0,1],\mathfrak{H}^j).$$

Consider the diagram

$$\mathfrak{H}^{j+1} \xrightarrow{\nabla_{t\,j+1}^{\,j}} L^2([0,1],\mathfrak{H}^j) \xrightarrow{E^i_j} L^2([0,1],\mathfrak{H}^i) \xrightarrow{\mathcal{S}_{t\,i}^{\,i+1}} \mathfrak{H}^{i+1}$$

The transformations are all bounded so that $M_t(E:d\Lambda)^{i+1}_{j+1} = \mathcal{S}_{t\,i}^{\,i+1}E^i_j\nabla_{t\,j+1}^{\,j}$ is a bounded linear transformation and (40) defines a cmx process $M(E:d\Lambda)$.

Since $\|\nabla_{t\,j+1}^{\,j}\| \le (j+1)^{1/2}$ it follows that $s \mapsto F_{s\,j}^{\,i}\nabla_{t\,j+1}^{\,j}$ is in the Lebesgue space $L^2([0,1],\mathcal{B}(\mathfrak{H}^{j+1},\mathfrak{H}^i)_{\mathrm{so}})$ and (41) defines a cmx process $M(F:dA)$.

If $\psi \in \mathfrak{H}^j$ then $s \mapsto G_{s\,j}^{\,i}\psi$ is a process $G^i_j\psi$ in $L^2([0,1],\mathfrak{H}^i)$ and $\psi \mapsto \mathcal{S}_{t\,i}^{\,i+1}G^i_j\psi$ is a linear transformation $\mathcal{S}_{t\,i}^{\,i+1}G$ in $\mathcal{B}(\mathfrak{H}^j,\mathfrak{H}^{j+1})$. Therefore (42) defines a cmx process $M(dA^\dagger:G)$.

To avoid reindexing we define $E^i_j = F^i_j = G^i_j = 0$ whenever $i < 0$ or $j < 0$.

The following theorem shows that the integrals defined above belong to $\mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and that it is natural to choose the respective integrands in the spaces $\mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, $\mathcal{L}^2_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, $\mathcal{L}^2_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\mathcal{L}^1_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$.

Theorem 6.1 *The formulae* (40), (41), (42) *and* (43) *define processes in* $\mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$. *Moreover*

$$\begin{aligned}
(M_t(E : d\Lambda)^{i+1}_{j+1})^* &= M_t(E^* : d\Lambda)^{j+1}_{i+1} && (44)\\
(M_t(F : dA)^{i}_{j+1})^* &= M_t(dA^\dagger : F^*)^{j+1}_{i}, && (45)\\
(M_t(H : dt)^{i}_{j})^* &= M_t(H^* : dt)^{j}_{i}, && (46)\\
\left\|M(E : d\Lambda)^{i+1}_{j+1}\right\|_\infty &\le (i+1)^{\frac{1}{2}}(j+1)^{\frac{1}{2}}\left\|E^i_j\right\|_\infty, && (47)\\
\left\|M(F : dA)^{i}_{j+1}\right\|_\infty &\le (j+1)^{\frac{1}{2}}\left\|F^i_j\right\|_2, && (48)\\
\left\|M(H : dt)^{i}_{j}\right\|_\infty &\le \left\|H^i_j\right\|_1. && (49)
\end{aligned}$$

Proof. The measurability of the processes follows from the measurability of $t \mapsto \nabla(t)$ and $t \mapsto \mathcal{S}_t$ and an extension of Corollary 2.3. It follows from (19) that

$$\left\|M(E : d\Lambda)^{i+1}_{j+1}\right\|_\infty \le \sup_{t\in[0,1]} \left\|\mathcal{S}_{t}{}^{i+1}_{i}\right\| \left\|E_{t}{}^{i}_{j}\right\| \left\|\nabla_{t}{}^{j}_{j+1}\right\| = (i+1)^{\frac{1}{2}}(j+1)^{\frac{1}{2}}\left\|E^i_j\right\|_\infty,$$

proving (47). If $\psi \in \mathfrak{H}^{j+1}$ then

$$\begin{aligned}
\left\|M_t(F : dA)^{i}_{j+1}\psi\right\| &\le \int_0^t \left\|F_{s}{}^{i}_{j}\right\| \left\|\nabla_{t}{}^{j}_{j+1}(s)\psi\right\| ds\\
&\le \left(\int_0^1 \left\|F_{s}{}^{i}_{j}\right\|^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 \left\|\nabla_{t}{}^{j}_{j+1}(s)\psi\right\|^2 ds\right)^{\frac{1}{2}}\\
&\le (j+1)^{\frac{1}{2}}\left\|F^i_j\right\|_2 \|\psi\|,
\end{aligned}$$

proving (48). A similar argument gives (49).

Suppose $f, g \in L^2(0,1)$. It follows from (21) that

$$\begin{aligned}
\left\langle M_t(E : d\Lambda)^{i+1}_{j+1} e^{j+1}(f), e^{i+1}(g)\right\rangle &= \left\langle \mathcal{S}_{t}{}^{i+1}_{i} E^i_j \nabla_{t}{}^{j}_{j+1} e^{j+1}(f), e^{i+1}(g)\right\rangle\\
&= \int_0^1 \left\langle E_{s}{}^{i}_{j} \nabla_{t}{}^{j}_{j+1}(s) e^{j+1}(f), \nabla_{t}{}^{i}_{i+1}(s) e^{i+1}(g)\right\rangle ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t f(s)\overline{g(s)} \left\langle E_s{}^i_j e^j(f), e^i(g) \right\rangle ds && (50)\\
&= \int_0^t f(s)\overline{g(s)} \left\langle e^j(f), (E^*)_s{}^j_i e^i(g) \right\rangle ds \\
&= \int_0^1 \left\langle \nabla_t{}^j_{j+1}(s) e^{j+1}(f), (E^*)_s{}^j_i \nabla_t{}^i_{i+1}(s) e^{i+1}(g) \right\rangle ds \\
&= \left\langle e^{j+1}(f), \mathcal{S}_t{}^{j+1}_j (E^*)^j_i \nabla_t{}^i_{i+1} e^{i+1}(g) \right\rangle ds \\
&= \left\langle e^{j+1}(f), M_t(E^* : d\Lambda)^{j+1}_{i+1} e^{i+1}(g) \right\rangle .
\end{aligned}
$$

Since $\{e^n(f) : f \in L^2[0,1]\}$ is total in $\mathfrak{H}^n$ and $M_t(E : d\Lambda)^{i+1}_{j+1}$ is bounded this proves (44). Put $G = F^*$. It follows from (21) that

$$
\begin{aligned}
\left\langle M_t(dA^\dagger : G)^{i+1}_{j+1} e^{j+1}(f), e^{i+1}(g) \right\rangle &= \left\langle \mathcal{S}_t{}^{i+1}_i G^i_{j+1} e^{j+1}(f), e^{i+1}(g) \right\rangle \\
&= \int_0^1 \left\langle G_s{}^i_{j+1} e^{j+1}(f), \nabla_t{}^i_{i+1}(s) e^{i+1}(g) \right\rangle ds \\
&= \int_0^t \overline{g(s)} \left\langle G_s{}^i_{j+1} e^{j+1}(f), e^i(g) \right\rangle ds && (51)\\
&= \int_0^t \overline{g(s)} \left\langle e^{j+1}(f), F_s{}^{j+1}_i e^i(g) \right\rangle ds && (52)\\
&= \int_0^1 \left\langle e^{j+1}(f), F_s{}^{j+1}_i \nabla_t{}^i_{i+1}(s) e^{i+1}(g) \right\rangle ds \\
&= \left\langle e^{j+1}(f), M_t(F : dA)^{j+1}_{i+1} e^{i+1}(g) \right\rangle ,
\end{aligned}
$$

and (45) follows as above. A similar argument gives (46). ∎

The *cmx quantum stochastic integral*, $M = M(E, F, G, H)$, of an integrable quadruple (E, F, G, H) of chaos matrices is the process

$$
M_t = M_t(E : d\Lambda) + M_t(F : dA) + M_t(dA^\dagger : G) + M_t(H : dt). \qquad (53)
$$

The term cmx quantum stochastic integral may also be used for the definite integral.

M is *regular* if $M_t \in \mathcal{B}_{\mathrm{cmx}}(\mathfrak{H})$ for $t \in [0,1]$. We shall us the notation

$$
M_t = \int_0^t (E_s \, d\Lambda_s + F_s \, dA_s + dA^\dagger_s G_s + H_s \, ds)
$$

although the subscript $_s$ in the integrand is sometimes suppressed. If $0 \leq t_1, t_2 \leq 1$ define,

$$\int_{t_1}^{t_2} (E\, d\Lambda + F\, dA + \, dA^\dagger G + H\, ds) = M_{t_2}(E,F,G,H) - M_{t_1}(E,F,G,H).$$

The integral behaves in behaves in the usual way with respect to t_1 and t_2.

The *control matrix* of M (of the integrable quadruple (E,F,G,H)) is the scalar matrix $\mathcal{K} = \mathcal{K}(M)$ with entries

$$\mathcal{K}^i_j = i^{1/2} \left\| E^{i-1}_{j-1} \right\|_\infty j^{1/2} + \left\| F^i_{j-1} \right\|_2 j^{1/2} + i^{1/2} \left\| G^{i-1}_j \right\|_2 + \left\| H^i_j \right\|_1, \qquad (54)$$

with the convention that $E^i_j = F^\nu_j = G^i_\nu = 0$ if $i < 0$ or $j < 0$. It follows from (47), (48) and (49) that

$$\left\| M^i_j \right\|_\infty \leq \mathcal{K}^i_j \qquad i,j = 0,1,2,\ldots.$$

Corollary 6.2 *If $\mathcal{K}$ is the cmx representation of a bounded operator in ℓ^2 then $M = \{M_t : t \in [0,1]\}$ is a regular cmx process.*

The following corollary to Theorem 6.1 is a chaos matrix version of [11, Theorems 4.1, 4.4] and is used to prove the uniqueness of cmx quantum stochastic integrals. It may be shown that, for adapted integrands, the definition of cmx quantum stochastic integral is compatible with Hudson and Parthasarathy's definition of quantum stochastic integral [11].

Corollary 6.3 *If $M = M(E,F,G,H)$ is the cmx quantum stochastic integral of an integrable quadruple of cmx processes and $f, g \in L^2[0,1]$ then*

$$\begin{aligned} \left\langle M_t{}^{i+1}_{j+1} e^{j+1}(f), e^{i+1}(g) \right\rangle \;&= \qquad (55) \\ &\int_0^t \Big(f(s)\overline{g(s)} \left\langle E_s{}^i_j e^j(f), e^i(g) \right\rangle + f(s) \left\langle F_s{}^{i+1}_j e^j(f), e^{i+1}(g) \right\rangle \\ &\quad + \overline{g(s)} \left\langle G_s{}^i_{j+1} e^{j+1}(f), e^i(g) \right\rangle + \left\langle H_s{}^{i+1}_{j+1} e^{j+1}(f), e^{i+1}(g) \right\rangle \Big)\, ds \end{aligned}$$

Proof. . The terms on the right involving E, F and G follow from (50), (51) and (52). The remaining term is immediate from the definition of $M(H : dt)$. ∎

Corollary 6.4 *The mapping $t \mapsto {M_t}^{i+1}_{j+1}$ is strongly continuous and strongly measurable.*

Proof. Since $\left\| {M_t}^{i+1}_{j+1} \right\| \leq \mathcal{K}^{i+1}_{j+1}$ it follows from Lebesgue's dominated convergence theorem that $t \mapsto \left\langle {M_t}^{i+1}_{j+1} e^{j+1}(f), e^{i+1}(g) \right\rangle$ is continuous whenever $f, g \in L^2[0,1]$. The conclusion follows since $\{e^k(f) : f \in L^2[0,1]\}$ is total in $\mathfrak{H}^k$ for $k = 0, 1, 2, \ldots$. ∎

The following Proposition extends the uniqueness theorems of Parthasarathy [21] Vincent–Smith [27] and Lindsay [15] to cmx quantum stochastic integrals.

Proposition 6.5 *If (E, F, G, H) is an integrable quadruple and*

$$\int_0^t (E\, d\Lambda + F\, dA + G\, dA^\dagger + H\, ds) = 0$$

then $E \equiv F \equiv G \equiv H \equiv 0$, the zero process.

Proof. We follow Lindsay's line of argument.

Let $L^{step}_{\mathbb{Q}} = L^{step}_{\mathbb{Q}}[0,1]$ be the vector space, over $\mathbb{Q}$, of $\mathbb{Q}$-valued step functions with discontinuity points in $\mathbb{Q}$ and, for $s \in [0,1]$, let $L^{step}_{\mathbb{Q}}(\hat{s}) = \{f \in L^{step}_{\mathbb{Q}} : f = 0 \text{ in a neighbourhood of } s\}$. Then both $\{e^j(f) : f \in L^{step}_{\mathbb{Q}}(\hat{s})\}$ and $\{e^j(f) : f \in L^{step}_{\mathbb{Q}} \setminus L^{step}_{\mathbb{Q}}(\hat{s})\}$ are total subsets of $\mathfrak{H}^j$.

The integrand in (55) is zero almost everywhere for each f, g in the countable set $L^{step}_{\mathbb{Q}}$. Therefore there exists a null set $\mathcal{N}$ such that

$$\begin{aligned} f(s)\overline{g(s)} \left\langle {E_s}^i_j e^j(f), e^i(g) \right\rangle + f(s) \left\langle {F_s}^{i+1}_j e^j(f), e^{i+1}(g) \right\rangle & \\ + \overline{g(s)} \left\langle {G_s}^i_{j+1} e^{j+1}(f), e^i(g) \right\rangle + \left\langle {H_s}^{i+1}_{j+1} e^{j+1}(f), e^{i+1}(g) \right\rangle &= 0 \end{aligned}$$

whenever $s \in [0,1] \setminus \mathcal{N}$, for all $f, g \in L^{step}_{\mathbb{Q}}$ and $i, j = 0, 1, 2, \ldots$.

Fix $s \in [0,1] \setminus \mathcal{N}$. Then $\left\langle {H_s}^{i+1}_{j+1} e^{j+1}(f), e^{i+1}(g) \right\rangle = 0$ whenever $f, g \in L^{step}_{\mathbb{Q}}(\hat{s})$. Since ${H_s}^{i+1}_{j+1}$ is bounded it follows that ${H_s}^{i+1}_{j+1} = 0$ for all i, j. Therefore $H \equiv 0$.

Similarly $f(s)\left\langle F_s{}^{i+1}_{j} e^j(f), e^{i+1}(g)\right\rangle = 0$ whenever $f \in L^{step}_{\mathbb{Q}}$ and $g \in L^{step}_{\mathbb{Q}}(\hat{s})$. Since $\{e^i(g) : g \in L^{step}_{\mathbb{Q}}(\hat{s})$ is total in $\mathfrak{H}^j$ it follows that $F_s{}^i_j e^j(f) = 0$ whenever $f \in L^{step}_{\mathbb{Q}} \setminus L^{step}_{\mathbb{Q}}(\hat{s})$. These $e^j(f)$ are total in $\mathfrak{H}^j$ and $F_s = 0$. Therefore $F \equiv 0$ and, taking adjoints, $G \equiv 0$.

Finally $\left\langle E_s{}^i_j e^j(f), e^i(g)\right\rangle = 0$ whenever $f, g \in L^{step}_{\mathbb{Q}} \setminus L^{step}_{\mathbb{Q}}(\hat{s})$ and, as in the other cases, $E \equiv 0$. ∎

We shall also need the following convergence theorem.

Theorem 6.6 *Let (E_n, F_n, G_n, H_n) be a sequence of integrable quadruples of cmx processes and suppose E_n, F_n, G_n and H_n converge to E, F, G, H respectively. Then $M(E_n, F_n, G_n, H_n)$ converges to $M(E, F, G, H)$*

Proof. This follows directly from Theorem 6.1. For example, for all i, j,

$$\lim_{n\to\infty} \left\| M(E_n - E : d\Lambda)^i_j \right\|_\infty \leq \lim_{n\to\infty} i^{1/2} j^{1/2} \left\| E^{i-1}_{j-1} - E_n{}^{i-1}_{j-1} \right\|_\infty = 0,$$

by the inequality (47). The inequalities (48) and (49) may be similarly used to prove the convergence in $\mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ of the sequences $M(F_n - F : dA)$, $M(dA^\dagger : G_n - G)$ and $M(H_n - H : dt)$ to the zero process. ∎

Theorem 6.7 *If (E, F, G, H) is an adapted integrable cmx quadruple then the cmx quantum stochastic integral $M(E, F, G, H)$ is an adapted cmx process.*

Proof. The cmx quantum stochastic integrals are given by (40), (41), (42), (43) and (53). Consider first $M = M(F : dA)$ given by (41). If $t < s$ then

$$M_t \nabla(s) = \int_0^t F_u \nabla(u) \nabla(s)\, du = \int_0^t F_u \nabla(s) \nabla(u)\, du = \int_0^t \nabla(s) F_u \nabla(u)\, du$$

$$= \nabla(s) \int_0^t F_u \nabla(u)\, du = \nabla(s) M_t.$$

These identities may be verified by using the entry-wise definition of M in (41) and the norm bounds (48) and computing entry-wise. In particular the norm bounds on the entries allow the interchange of $\nabla(s)$ and the integral over $[0, t]$.

If $\psi \in L^2([0,1], \mathfrak{H}^j)$ and $t < s$ then

$$\begin{aligned}
\nabla(s)(\mathcal{S}_t\psi)(x_1,\ldots,x_{j+1}) &= (j+1)^{\frac{1}{2}}(\mathcal{S}_t\psi)(x_1,\ldots,x_j,s) \\
&= \sum_{k=0}^{j} \chi_{[0,t]}(x_k)\psi(x_k)(x_1,\ldots,\hat{x}_k,\ldots,x_j,s) \\
&\qquad +\chi_{[0,t]}(s)\psi(s)(x_1,\ldots,x_k,\ldots,x_j) \\
&= j^{-\frac{1}{2}}\sum_{k=0}^{j} \chi_{[0,t]}(x_k)(\nabla(s)\psi)(x_k)(x_1,\ldots,\hat{x}_k,\ldots,x_j) \\
&= (\mathcal{S}_t\nabla(s)\psi)(x_1,\ldots,x_j).
\end{aligned}$$

Therefore $\nabla(s)\mathcal{S}_t = \mathcal{S}_t\nabla(s)$ whenever $t < s$.

Now $M^* = M(dA^\dagger : G)$ where $G = F^*$. Since F is adapted so is G.

$$M_t^*\nabla(s) = \mathcal{S}_t G_t \nabla(s) = \mathcal{S}_t\nabla(s)G_t = \nabla(s)\mathcal{S}_t G_t = \nabla(s)M_t^*$$

whenever $t < s$ and M is adapted. Since $M(dA^\dagger : G)^* = M(G^* : dA)$ it follows that $M(dA^\dagger : G)$ is adapted.

If $M = M(E : d\Lambda)$ then, since E is adapted

$$M_t\nabla(s) = \mathcal{S}_t E_t \nabla_t \nabla(s) = \mathcal{S}_t E_t \nabla(s)\nabla_t = \nabla(s)\mathcal{S}_t E_t \nabla_t = \nabla(s)M_t$$

whenever $t < s$. Since $E = E^*$ it follows that $M = M^*$ so that M is adapted. We omit the proof that $M(H : dt)$ is adapted. ∎

Although the natural integrands for quantum stochastic integrals are integrable quadruples of operator processes the processes are more amenable when the operators are bounded. For example the quantum semimartingales of Attal and Meyer [3] have integrands in the Lebesgue spaces $L^p([0,1], \mathcal{B}(\mathfrak{H})_{\mathrm{so}})$ and the quantum Ito formula [28] is valid for regular quantum semimartingales.

A *quantum semimartingale* (non-commutative semimartingale [1, §II]) is an adapted quantum stochastic process $\hat{M} = \{\hat{M}_t : t \in [0,1]\}$ with representation as a Hudson–Parthasarthy quantum stochastic integral

$$\hat{M}_t = M_0 + \int_0^t (\hat{E}_s\, d\Lambda_s + \hat{F}\, dA + dA^\dagger\, \hat{G} + \hat{H}_s\, ds), \qquad t \in [0,1], \tag{56}$$

where the adapted processes $(\hat{E}, \hat{F}, \hat{G}, \hat{H})$ with

$$\hat{E} \in L^\infty([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}}), \quad \hat{F}, \hat{G} \in L^2([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}}),$$

$$\hat{H} \in L^1([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}}), \tag{57}$$

form an *integrable quadruple*.

If there is no initial space $\hat{M}_0$ is a multiple of the identity. Otherwise $\hat{M}_0$ is the ampliation of an operator in the initial space. It is shown in [16, §6], [11, §4] that $\hat{M}_t$ is defined as an operator in $\mathcal{H}$ with domain $\mathcal{E}$ for each $t \in [0,1]$. Sometimes $\hat{M}_t$ is extended to its maximal domain [3]. The representation (56) of $\hat{M}$ is unique.

If $\hat{E}_t = (\hat{E}_t)^*$, $\hat{G}_t = (\hat{F}_t)^*$ and $\hat{H}_t = (\hat{H}_t)^*$ for almost all $t \in [0,1]$ then $\hat{M}$ is said to be *symmetric*. In this case $\hat{M}_t$, with domain $\mathcal{E}$, is a symmetric operator for each $t \in [0,1]$.

If $\hat{M}$ is symmetric and the operators $\hat{M}_t$ are essentially self-adjoint on a common core for almost all $t \in [0,1]$ then $\hat{M}$ is *essentially self-adjoint*.

If $\hat{M}_t$ is a bounded operator then $\hat{M}_t \in \mathcal{B}(\mathfrak{H})$ will always denote the unique extension of $\hat{M}_t$ to $\mathfrak{H}$. If $\hat{M} \in L^\infty([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}})$ then the quantum semimartingale $\hat{M}$ is said to be *regular*. If $\hat{M}$ is regular and symmetric then $\hat{M}$ is said to be *self-adjoint*.

The vector space of quantum semimartingales is denoted $\mathcal{S}'$ and that of regular quantum semimartingales is denoted $\mathcal{S}$.

A *cmx semimartingale* is a cmx quantum stochastic integral

$$M_t = M_0 + \int_0^t (E_s\, d\Lambda_s + F_s\, dA_s + G_s\, dA_s^\dagger + H_s\, ds), \qquad t \in [0,1], \tag{58}$$

where (E, F, G, H) is an adapted integrable quadruple with .

$$E \in L^\infty_{\text{cmx}}([0,1], \mathcal{B}_{\text{cmx}}(\mathfrak{H})_{\text{so}}), \quad F, G \in L^2_{\text{cmx}}([0,1], \mathcal{B}_{\text{cmx}}(\mathfrak{H})_{\text{so}}),$$
$$H \in L^1_{\text{cmx}}([0,1], \mathcal{B}_{\text{cmx}}(\mathfrak{H})_{\text{so}})$$

It follows from Proposition 6.5 and Theorem 6.7 that M is adapted and well defined by (58).

If (E, F, G, H) is symmetric M is said to be *symmetric*.

If M is regular it follows from Corollary 6.4 (ii) that $M \in L^\infty_{\text{cmx}}([0,1], \mathcal{B}_{\text{cmx}}(\mathfrak{H})_{\text{so}})$. If M is regular and symmetric then M is said to be *self-adjoint*.

The vector spaces of quantum semimartingales and cmx semimartingales are denoted $\mathcal{S}'$ and $\mathcal{S}'_{\text{cmx}}$. The vector spaces of regular quantum semimartingales and regular cmx semimartingales are denoted $\mathcal{S}$ and $\mathcal{S}_{\text{cmx}}$.

Theorem 6.8 *Let $\hat{M}$ be a quantum semimartingale*

$$\hat{M}_t = \int_0^t (\hat{E}\, d\Lambda + \hat{F}\, dA + \hat{G}\, dA^\dagger + \hat{H}\, ds) \tag{59}$$

and let M be the cmx semimartingale,

$$M_t = \int_0^t (E\, d\Lambda + F\, dA + G\, dA^\dagger + H\, ds),$$

where E, F, G, H are the cmx representations of $\hat{E}, \hat{F}, \hat{G}, \hat{H}$ respectively. Then

(i) *M is a cmx representation of $\hat{M}$ with $\mathcal{E} \subset \mathfrak{D}(M_t)$ for all $t \in [0,1]$.*

(ii) *The mapping $M \mapsto \hat{M}$ is a bijection from $\mathcal{S}'_{\text{cmx}}$ onto $\mathcal{S}'$.*

(iii) *The mapping $M \mapsto \hat{M}$ is an isometric isomorphism from $\mathcal{S}_{\text{cmx}}$ onto $\mathcal{S}$.*

Proof. It follows from Proposition 2.6 and Corollary 4.4 that the formula (59) defines a quantum semimartingale $\hat{M}$. If $t \in [0,1]$ then $\hat{M}_t$ is characterised in [11] and [16] as the unique operator in $\mathcal{L}(\mathfrak{H})$ with domain $\mathcal{E}$ such that, whenever $f, g \in L^2[0,1]$,

$$\left\langle \hat{M}_t e(f), e(g) \right\rangle = \int_0^t \left\langle (f\bar{g}\hat{E} + f\hat{F} + \bar{g}\hat{G} + \hat{H})(s) e(f), e(g) \right\rangle ds.$$

We shall only consider the case $\hat{F} = \hat{G} = \hat{H} = 0$. The proofs of the three other cases with a single non-zero integrand are simple variants.

If $g, h \in L^2[0,1]$ then

$$\begin{aligned}
\sum_{i,j=0}^{\infty} \left\langle g(s) E_s{}^i_j e^j(g), h(s) e^i(h) \right\rangle &\leq |g(s)h(s)| \sum_{i,j=0}^{\infty} \left\| E^i_j \right\|_\infty | \left\langle e^j(g), e^i(h) \right\rangle | \\
&\leq |g(s)h(s)| \left\| \tilde{E} \right\|_\infty \langle e(|g|), e(|h|) \rangle , \qquad (60)
\end{aligned}$$

which is an integrable function.

$$\begin{aligned}
\left\langle \hat{M}_t e(g), e(h) \right\rangle &= \int_0^t \left\langle g(s) \hat{E}_s e(g), h(s) e(h) \right\rangle ds \\
&= \int_0^t \left\langle g(s) \sum_{j=0}^{\infty} E_s{}^i_j e^j(g), h(s) \sum_{i=0}^{\infty} e^i(h) \right\rangle ds \qquad (61) \\
&= \sum_{i,j=0}^{\infty} \int_0^t \left\langle g(s) E_s{}^i_j e^j(g), h(s) e^i(h) \right\rangle ds \qquad (62) \\
&= \sum_{i,j=0}^{\infty} \int_0^t \left\langle E_s{}^i_j \nabla_s e^{j+1}(g), h(s) \nabla_s e^{i+1}(h) \right\rangle ds \\
&= \sum_{i,j=0}^{\infty} \left\langle M_t{}^i_j e^j(g), e^i(h) \right\rangle . \qquad (63)
\end{aligned}$$

(61) follows from the absolute convergence of the left hand series in (60). The third equality is a consequence of Lebesgue's dominated convergence theorem gives (62) and (63) follows from the definition (40) and (21). Now

$$\left\| M_t{}^i_j e^j(g) \right\| \leq i^{\frac{1}{2}} j^{\frac{1}{2}} \left\| E \right\|_\infty \frac{\|g\|^j}{(j!)^{\frac{1}{2}}} \leq i^{\frac{1}{2}} \left\| E \right\|_\infty \frac{\|g\|^j}{((j-1)!)^{\frac{1}{2}}}$$

and $\sum_{j=0}^{\infty} M_t{}^i_j e^j(g)$ converges to η^i in $\mathfrak{H}^i$. Therefore, if $z \in \mathbb{C}$,

$$\begin{aligned}
\left\langle \hat{M}_t e(g), e(zh) \right\rangle &= \sum_{i,j=0}^{\infty} z^i \left\langle M_t{}^i_j e^j(g), e^i(h) \right\rangle \\
&= \sum_{i=0}^{\infty} z^i \left\langle \eta^i, e^i(h) \right\rangle ; \\
\left\langle \hat{M}_t e(g), e(zh) \right\rangle &= \sum_{i=0}^{\infty} z^i \left\langle (\hat{M}_t e(g))^i, e^i(h) \right\rangle ,
\end{aligned}$$

and

$$\sum_{i=0}^{\infty} z^i \left\langle (\hat{M}_t e(g))^i - \eta^i, e^i(h) \right\rangle = 0.$$

Therefore $\left\langle (\hat{M}_t) e(g))^i - \eta^i, e^i(h) \right\rangle = 0$ for all $h \in L^2[0,1]$. Since the exponential domain is dense in $\mathfrak{H}$ it follows that $(\hat{M}_t e(g))^i = \eta^i$ for all i and $\sum_{i=0}^{\infty} \eta^i$ is convergent in $\mathfrak{H}$ to $\hat{M}_t e(g)$. Therefore $e(g) \in \mathfrak{D}(M_t)$ and

$$\tilde{M}_t e(g) = M_t e(g) = \hat{M}_t e(g), \qquad t \in [0,1],\ g \in L^2[0,1],$$

Therefore M is a cmx representation of $\hat{M}$. This proves (i) in the case $F = G = H = 0$.

By linearity $M_t \psi = \tilde{M}_t \psi$ for all $\psi \in \mathcal{E}$ and M_t is a cmx representation of $\tilde{M}_t$.

It follows from Corollary 4.4 that $(E, F, G, H) \mapsto (\hat{E}, \hat{F}, \hat{G}, \hat{H})$, is a one to one correspondence between the integrands of $\mathcal{S}_{\text{cmx}}$ and the integrands of $\mathcal{S}'$. The one to one correspondence between M_t and $\hat{M}_t$ now follows from Proposition 6.5 and the uniqueness theorem for quantum stochastic integrals [15], [27, Theorem 4.7] ∎

The quantum Ito product formula [11, Theorem 4.5], its extension [3, Theorem 4], and the functional quantum Ito formulae [28, Theorem 4.2, Theorem 6.2.] are true for regular quantum semimartingales. Using the isometric isomorphism $M \mapsto \tilde{M}$ of $\mathcal{S}_{\text{cmx}}$ and $\mathcal{S}$ in Theorem 6.8 the corresponding Ito formulae are also valid for regular cmx semimartingales.

We will reverse this procedure. The functional Ito formula will be proved for a class of process $M \in \mathcal{S}_{\text{cmx}}$. The corresponding formula for the Hudson–Parthasarathy process $\tilde{M}$ will be deduced from the correspondence between quantum semimartingales and cmx semimartingales in Theorem 6.8.

We conclude this section with a sufficient condition that M be a regular cmx process.

Theorem 6.9 *If E, F, G, H are cmx processes such that $(N+I)^{1/2} E (N+I)^{1/2}$ is in $L^{\infty}_{\text{cmx}}(\mathfrak{H})_{\text{so}}$, $F(N+I)^{1/2}$ and $(N+I)^{1/2} G$ are in $L^2_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ and H is in $L^1_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ then $M(E, F, G, H)$ is in $L^{\infty}_{\text{cmx}}(\mathfrak{H})_{\text{so}}$.*

Proof. The conditions automatically imply that the cmx quantum stochastic integral

$$M_t = \int_0^t (E\, d\Lambda + F\, dA + G\, dA^\dagger + H\, ds)$$

exists. It follows from (19) that $\mathcal{S}_t(N+I)^{-1/2}$ and $(N+I)^{-1/2}\nabla_t$ represent contractions. It follows from (47) that the operator

$$M_t(E : d\Lambda) = \mathcal{S}_t E \nabla_t = \mathcal{S}_t(N+I)^{-1/2}(N+I)^{1/2}E(N+I)^{1/2}(N+I)^{-1/2}\nabla_t$$

is bounded in norm by $\leq \left\|(N+I)^{1/2}E(N+I)^{1/2}\right\|_\infty$.

Similarly, if $\psi \in \mathfrak{H}^0 \oplus \mathfrak{H}^1 \oplus \cdots \oplus \mathfrak{H}^j$ for some j then

$$\begin{aligned}
\|M_t(F : dA)\psi\| &= \left\|\int_0^1 F_s(N+I)^{1/2}(N+I)^{-1/2}\nabla_t(s)\psi\right\| ds \\
&\leq \int_0^1 \left\|F_s(N+I)^{1/2}\right\| \left\|(N+I)^{-1/2}\nabla_t(s)\psi\right\| ds \\
&\leq \left\|F(N+I)^{1/2}\right\|_2 \|\psi\| .
\end{aligned}$$

Since such ψ are dense in $\mathfrak{H}$ it follows that $M(F : dA)$ is in $L^\infty_{\text{cmx}}(\mathfrak{H})_{\text{so}}$.

If $\psi = \psi^0 + \psi^1 + \ldots + \psi^k$ with $\psi^j \in \mathfrak{H}^j$ then

$$\begin{aligned}
\left\|M_t(dA^\dagger : G)\psi\right\|^2 &= \left\|\mathcal{S}_t(N+I)^{1/2}\right\| \left\|(N+I)^{-1/2}G\psi\right\|^2 \\
&= \sum_{j=0}^k \int_0^t \left(G_s\psi^j(x_1,\ldots,x_j)\right)^2 \, ds\, dx_1 \ldots dx_j \\
&\leq \sum_{j=0}^k \int_0^T \|G_s\|^2 \left|\psi^j(x_1,\ldots,x_j)\right|^2 ds\, dx_1 \ldots dx_j \\
&\leq \|G\|_2^2 \|\psi\|^2 ,
\end{aligned}$$

and $M(d\Lambda : G)$ is in $L^\infty_{\text{cmx}}(\mathfrak{H})_{\text{so}}$. The remaining case is straightforward. ∎

It is now easy to find bounded cmx quantum stochastic integrals. If E, F, G, H are as in Theorem 6.9 and

$$\breve{E} = (N+1)^{-\frac{1}{2}}E(N+1)^{-\frac{1}{2}}, \quad \breve{F} = F(N+1)^{-\frac{1}{2}}, \quad \breve{G} = (N+1)^{-\frac{1}{2}}G$$

then $\breve{E} \in L^\infty_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ and $\breve{F},\ \breve{G} \in L^2_{\text{cmx}}(\mathfrak{H})_{\text{so}}$. It follows from Theorem 6.9 that $M(\breve{E}, \breve{F}, \breve{G}, H)$ belongs to $L^\infty_{\text{cmx}}(\mathfrak{H})_{\text{so}}$.

The use of Theorem 6.9 in the hunt for sufficient conditions on its integrands for a quantum semimartingale to be regular may be a chimera. This is because the number operator is not adapted so that the quadruple $(\breve{E}, \breve{F}, \breve{G}, H)$ will not, normally, be adapted. Similar difficulties are encountered if the number operator N is replaced by the adapted number process $\{N_{t]} \otimes I_{(t}$, where $N_{t]}$ is the number operator for $\mathfrak{H}_{t]}$.

7 Scalar Matrices

We recall Nelson's analytic vector theorem [17]. A proof of the theorem may be found in [30, §8.5]. If $\hat{T}$ is a self-adjoint operator in a Hilbert space $\mathcal{H}$ then $C^\infty(\hat{T}) = \bigcap_{k=0}^\infty \mathcal{D}(\hat{T}^k)$. An element $\varphi \in C^\infty(\hat{T})$ is an analytic vector of $\hat{T}$ with radius of convergence $r(\varphi)$ if r is the radius of convergence of the complex power series

$$\sum_{k=0}^\infty \frac{\|\hat{T}^k\varphi\|}{k!} z^k$$

The set of analytic vectors of $\hat{T}$ is denoted $\mathcal{A}(\hat{T})$ and $\mathcal{A}_r(\hat{T}) = \{\varphi \in \mathcal{A}(\hat{T}) : r(\varphi) \geq r\}$.

Theorem 7.1 (Nelson) *Let $\hat{T}$ be a symmetric operator on a Hilbert space $\mathcal{H}$ whose domain $\mathcal{D}(\hat{T})$ contains a dense subspace $\mathcal{D}$ of analytic vectors. Then*

(i) *$\hat{T}$ is essentially self-adjoint.*

(ii) *$\mathcal{D}$ is a core for $\hat{T}$.*

(iii) *If φ is an analytic vector of $\hat{T}$ and $p \in \mathbb{C}$ then*

$$e^{ip\overline{T}}\varphi = \sum_{k=0}^\infty \frac{p^k}{k!}\hat{T}^k\varphi \qquad \text{whenever } |p| < r(\varphi)$$

where $\overline{T}$ is the (self-adjoint) closure of $\hat{T}$.

Lemma 7.2 *Let $\mathcal{V} \succ 0$ be a scalar matrix. Then $\ell_{00} \subset C^\infty(\tilde{\mathcal{V}})$ if and only if $\ell_{00} \subset C^\infty(\mathcal{V})$.*

In this case $\mathcal{V}^k x = (\tilde{\mathcal{V}})^k x$ *whenever* $x \in \ell_{00}$.

Proof. It is enough to consider $x = e^j = (0, 0, \ldots, 0, 1, 0, \ldots)^t$, $j = 0, 1, \ldots$, which span ℓ_{00}.

Suppose $e^j \in C^\infty(\tilde{\mathcal{V}})$ for all j. A simple induction shows that, for each k,

$$\eta^{(k)}{}^i_j = \sum\{\mathcal{V}^i_{j_k}\mathcal{V}^{j_k}_{j_{k-1}} \cdots \mathcal{V}^{j_k}_j : (j, j_2, \ldots, j_k) \in \mathbb{N}^k\} \tag{64}$$

is convergent for all i and j, and that

$$(\tilde{\mathcal{V}})^k e^j = (\eta^{(k)}{}^0_j, \eta^{(k)}{}^1_j, \eta^{(k)}{}^i_j, \ldots)^t \in \ell^2.$$

This implies that $\mathcal{V}^k$ exists for all k and has entries $\eta^{(k)}{}^i_j$.

Since $\eta^{(k+1)}{}^i_j = \sum\{\mathcal{V}^i_{j_{k+1}}\eta^{(k)}{}^{j_{k+1}}_j : j_{k+1} = 0, 1, 2, \ldots\}$ it follows that $e^j \in \mathcal{D}(\mathcal{V}^k)$ for all k.

Conversely, if $\mathcal{V}^k$ exists for all k the formula (64) defines $\eta^{(k)}{}^i_j$ for all i, j and k. If $e^j \in C^\infty(\mathcal{V})$ then, for all k,

$$\mathcal{V}^k e^j = (\eta^{(k)}{}^0_j, \eta^{(k)}{}^1_j, \eta^{(k)}{}^i_j, \ldots)^t \in \ell^2.$$

Since $\eta^{(k+1)}{}^i_j = \sum\{\mathcal{V}^i_{j_{k+1}}\eta^{(k)}{}^{j_{k+1}}_j : j_{k+1} = 0, 1, 2, \ldots\}$ it follows that $\mathcal{V}^{k+1}e^j = \tilde{\mathcal{V}}(\mathcal{V}^k e^j)$ and $e^j \in C^\infty(\tilde{\mathcal{V}})$ for all j. ∎

Proposition 7.3 *Let* $\Xi \succ 0$ *be a (2k+1)-diagonal scalar matrix, let* $\xi > 0$ *and let* $\epsilon = \xi^{-1}(4k^2 + 2k)^{-1}$. *Then*

(i) *If* $\Xi^i_j \leq \xi(i + j)$ *for all* i, j *then* $\ell_{00} \subset \mathcal{A}_\epsilon(\Xi)$;

(ii) *If* $\Xi^i_j \leq \xi(i^{1/2} + j^{1/2})$ *for all* i, j *then* $\ell_{00} \subset \mathcal{A}_\infty(\Xi)$

Proof. (i) It is enough to consider the case $\xi = 1$. A simple induction shows that $x^{(n)} = \Xi^n \epsilon_j \in \ell_{00}$ with $x^{(n)}_r = 0$ for $r > j + nk$. Assume, inductively, that

$$x^{(n-1)}_r \leq (2j + k)(2j + 3k) \cdots (2j + (2n + 1)k)(2k + 1)^{n-1} \tag{65}$$

for all r. Now $x^{(n)} = \Xi x^{(n-1)}$ and, since Ξ is $2k + 1$ diagonal, the largest entry in Ξ which can multiply a non-zero element of $x^{(n-1)}$ is $\Xi^{j+nk}_{j+(n-1)k} \leq$

$2j + (2n+3)k$. Since Ξ is $2k+1$ diagonal at most $2k+1$ of the entries of $x^{(n-1)}$ contribute to each entry of $x^{(n)}$. Therefore

$$x_r^{(n)} \leq (2j+k)(2j+3k)\cdots(2j+(2n+3)k)(2k+1)^n$$

$$\frac{\|\Xi^n \epsilon_j\|}{n!} = \frac{\|x^{(n)}\|}{n!} \leq \frac{(2j+k)(2j+3k)\cdots(2j+(2n+3)k)(2k+1)^n(j+nk)^{\frac{1}{2}}}{n!} = c_n.$$

Now

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{(2j+(2n+5)k)(2k+1)(j+(n+1)k)^{\frac{1}{2}}}{(n+1)(j+nk)^{\frac{1}{2}}} = 2k(2k+1).$$

By the limit ratio test ϵ_j is an analytic vector for Ξ with $r(\epsilon_j) = 1/(4k^2 + 2k) = \epsilon$ for all j. Therefore $\ell_{00} \subset \mathcal{A}_\epsilon(\Xi)$.

(ii) Using the same argument as in (i) with the bound $\Xi^{j+nk}_{j+(n-1)k} \leq 2(j+(n+1)k)^{1/2}$ gives that

$$x_r^{(n-1)} \leq 2^{n-1}[(j+k)(j+2k)\cdots(j+nk)]^{\frac{1}{2}}(2k+1)^{n-1}.$$

The limit ratio test then shows that the power series $\sum_{n=0}^{\infty} z^n \|\Xi^n\epsilon_j\|/n!$ has infinite radius of convergence and $\epsilon_j \in \mathcal{A}_\infty(\Xi)$. Therefore $\ell_{00} \subset \mathcal{A}_\infty(\Xi)$. ∎

The following proposition follows directly from the definition of $\mathcal{K}$.

Proposition 7.4 *Let M be a cmx semimartingale, with control matrix $\mathcal{K}$, whose integrands E, F, G, H are $(2k+1)$-diagonal matrices. Suppose there exists a constant ξ such that*

$$\left\|E^i_j\right\|_\infty, \quad \left\|F^i_j\right\|_2, \quad \left\|G^i_j\right\|_2, \quad \left\|H^i_j\right\|_1 \leq \xi \tag{66}$$

for all $i, j = 0, 1, 2, \ldots$. If $\epsilon = \xi^{-1}(4(k+1)^2 + 2(k+1))^{-1}$ then

(i) $\ell_{00} \subset \mathcal{A}_\epsilon(\mathcal{K})$.

(ii) If $E = 0$ then $\ell_{00} \subset \mathcal{A}_\infty(\mathcal{K})$.

If K is a $(2k+1)$-diagonal process defined by (**??**) and with $\left\|{k_t}^i_j\right\| \leq \xi$ for all $t \in [0,1]$ then K satisfies (66). Thus there is a large class of cmx semimartingales whose control matrices satisfy the conclusions of Proposition 7.3. These control matrices represent essentially self-adjoint operators.

Proposition 7.5 *Let $\mathcal{V}$ be a scalar matrix with $\mathcal{V} \succ 0$, $\mathcal{V} = \mathcal{V}^*$.*

(i) *If* $\ell_{00} \subset \mathcal{A}(\mathcal{V})$ *then* $\tilde{\mathcal{V}}$ *is essentially self-adjoint, with closure* $\overline{\mathcal{V}}$.

(ii) *If* $\ell_{00} \subset \mathcal{A}_1(\mathcal{V})$ *and* $-1 \leq \rho \leq 1$ *then* $\sum_{k=0}^{\infty}(i\rho\mathcal{V})^k/k!$ *converges entrywise to the unique bounded scalar matrix* $e^{i\rho\mathcal{V}}$ *representing* $e^{i\rho\overline{\mathcal{V}}}$.

Proof. (i) By Lemma 7.2 if $x \in \ell_{00}$ then $x \in \mathcal{A}(\tilde{\mathcal{V}})$ and the two series

$$\sum_{k=0}^{\infty} \frac{i\rho^k}{k!}(\tilde{\mathcal{V}})^k x \quad = \quad \sum_{k=0}^{\infty} \frac{i\rho^k}{k!}\mathcal{V}^k x \tag{67}$$

are identical and absolutely convergent for $|\rho| \leq r(x)$. Therefore $\ell_{00} \subset \mathcal{A}(\tilde{\mathcal{V}})$. Since ℓ_{00} is dense in ℓ^2 it follows from Nelson's analytic vector theorem that $\tilde{\mathcal{V}}$ is essentially self-adjoint. It also follows that the sum of the series (67) is $e^{i\rho\overline{\mathcal{V}}}x$ for each $x \in \ell_{00}$

(ii) $\mathcal{V}^k e^j$ is the jth column vector in $\mathcal{V}^k$. Taking $x = e^j$, $j = 0, 1, 2, \ldots$ in (67) shows that the columns of $\sum_{k=0}^{\infty}(i\rho\mathcal{V})^k/k!$ sum in ℓ^2, and *a fortiori* coordinatewise to the corresponding column of the matrix representing the bounded linear transformation $e^{i\rho\overline{\mathcal{V}}}$. ∎

The *scalar matrix* of $X \in \mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ is the real matrix $\mathcal{V}(X)$ with entries

$$\mathcal{V}^i_j(X) = \left\| X^i_j \right\|_p, \qquad i, j = 0, 1, 2, \ldots.$$

Processes in $\mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ are controlled by their scalar matrices. The proof of the following lemma is an immediate consequence of the definitions.

Lemma 7.6 *Let* S_n *be a sequence in* $\mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$.

(i) S_n *is convergent to* S *in* $\mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *if and only if* $\mathcal{V}(S - S_n)$ *converges to the zero matrix.*

(ii) S_n *is absolutely summable if and only if* $\mathcal{V}(S_n)$ *is summable.*

Lemma 7.7 *Let* $p, q, r \in \{1, 2, \infty\}$ *with* $1/p + 1/q = 1/r$ *and let* $0 \prec \Phi$ *and* $0 \prec \Psi$ *be scalar matrices such that the product matrix* $\Phi\Psi$ *exists. If* $S \in \mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *and* $T \in \mathcal{L}^q_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *are such that* $\mathcal{V}(S) \prec \Phi$ *and* $\mathcal{V}(T) \prec \Phi$ *then the product process* ST *exists in* $\mathcal{L}^r_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *and* $\mathcal{V}(ST) \prec \Phi\Psi$.

Proof. The product $S^i_k T^k_j$ exists in $L^r([0,1], \mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i))$ and $\left\| S^i_k T^k_j \right\|_r \leq \boldsymbol{\nu}(S)^i_k \boldsymbol{\nu}(T)^k_j$. Therefore

$$\sum_{k=0}^{\infty} \left\| S^i_k T^k_j \right\|_r \leq \sum_{k=0}^{\infty} \boldsymbol{\nu}(S)^i_k \boldsymbol{\nu}(T)^k_j = (\boldsymbol{\nu}(S)\boldsymbol{\nu}(T))^i_j \leq (\Phi\Psi)^i_j < \infty$$

and the sequence $\{S^i_k T^k_j : k = 0, 1, \ldots\}$ is absolutely summable and therefore summable in $L^r([0,1], \mathcal{B}(\mathfrak{H}^j, \mathfrak{H}^i))$. Therefore the product process ST exists in $\mathcal{L}^r_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and is dominated by $\Phi\Psi$. ∎

If $\varphi \in \mathfrak{H}$ define $\nu(\varphi) \in \ell^2$ by the formula

$$\nu(\varphi) = (\|\varphi^0\|, \|\varphi^1\|, \ldots, \|\varphi^k\|, \ldots)^t$$

Lemma 7.8 *If $M \in \mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\boldsymbol{\nu}(M)^k$ exists then M^k_t exists and $\widetilde{M^k_t} = (\tilde{M}_t)^k$ for almost all $t \in [0,1]$.*

If $\psi \in \mathfrak{D}(M^k_t)$ then $\nu(M^k_t\psi) \prec \boldsymbol{\nu}(M)^k \nu(\psi)$ for almost all t.

Proof. It follows from Lemma 7.7 that M^k_t exists. Let $\boldsymbol{\nu} = \boldsymbol{\nu}(M)$. If $\psi \in \mathfrak{H}$ then

$$\left\| \sum_{j=0}^{\infty} M_t{}^i_j \psi^j \right\| \leq \sum_{j=0}^{\infty} \left\| M_t{}^i_j \right\| \left\| \psi^j \right\| \leq \sum_{j=0}^{\infty} \left\| \boldsymbol{\nu}^i_j \right\| \left\| \psi^j \right\|,$$

and $\nu(M_t\psi) \prec \boldsymbol{\nu}\nu(\psi)$. This may be iterated to give $\nu(M^k_t\psi) \prec \boldsymbol{\nu}(M)^k \nu(\psi)$. ∎

Lemma 7.9 *If $X \in \mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\psi \in \mathfrak{H}$ with $\nu(\psi) \in \mathcal{A}(\boldsymbol{\nu}(X))$ then $\psi \in \mathcal{A}(X_t)$ for almost all $t \in [0,1]$.*

Proof. Let $\boldsymbol{\nu} = \boldsymbol{\nu}(X)$. Since $C^\infty(\boldsymbol{\nu}) \neq \emptyset$ it follows by Lemma 7.7 that $\boldsymbol{\nu}^k$ and therefore X^k each exist for all k. Since $\boldsymbol{\nu}_t(X^k) \prec \boldsymbol{\nu}^k$ for almost all t it follows that $\psi \in \mathcal{A}(X_t)$ for almost all t. ∎

Theorem 7.10 *Let M be symmetric cmx process in $\mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ with scalar matrix $\boldsymbol{\nu} = \boldsymbol{\nu}(M)$. If $\ell_{00} \subset \mathcal{A}(\boldsymbol{\nu}(M))$ then*

(i) *$\tilde{M}$ is essentially self-adjoint with core $\mathfrak{H}_{00} \subset \mathcal{A}(M_t) \cap \mathcal{A}(\tilde{M}_t)$ for almost all $t \in [0,1]$*

(ii) *If* $\ell_{00} \subset \mathcal{A}_1(\boldsymbol{\nu}(M))$ *and* $\rho \in [-1,1]$ *then* $\sum_{k=0}^{\infty}(i\rho M)^k/k!$ *is convergent in* $\mathcal{L}^{\infty}_{\rm cmx}(\mathfrak{H})_{\rm so}$ *to a process* J.

For $t \in [0,1]$ *let* $\overline{M}_t$ *be the closure of* $\tilde{M}_t$. *Then* J_t *is the cmx representation of the unitary operator* $e^{i\rho\overline{M}_t}$.

Proof. By definition $\boldsymbol{\nu}^k$ exists and by Lemma 7.8 that M_t^k exists for all k.

(i) If $\psi \in \mathfrak{H}_{00}$ then $\nu(\psi) \in \ell_{00}$ and

$$\sum_{n=0}^{\infty} \frac{\rho^n}{n!} \|M_t^n \psi\| \leq \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \|\boldsymbol{\nu}^n \nu(\psi)\| < \infty \tag{68}$$

whenever $|\rho| < r(\nu(\psi))$. By Lemma 7.8 this remains true if M_t replaced by $\tilde{M}_t$. Therefore $\mathfrak{H}_{00} \subset \mathcal{A}(M_t) \cap \mathcal{A}(\tilde{M}_t)$ for all $t \in [0,1]$ and (i) follows from Theorem 7.1.

(ii) It follows from Proposition 7.5 that $\sum_{k=0}^{\infty}(\boldsymbol{\nu})^k/k!$ is entrywise convergent to a scalar matrix. By Lemma 7.7 $\boldsymbol{\nu}(M^k) \prec \boldsymbol{\nu}(M)^k = \boldsymbol{\nu}^k$ and, by Lemma 7.6, $\sum_{k=0}^{\infty}(i\rho M)^k/k!$ is convergent to a process J in $\mathcal{L}^{\infty}_{\rm cmx}(\mathfrak{H})_{\rm so}$.

If $\psi \in \mathfrak{H}_{00}$ and $|\rho| \leq 1$ then by Lemma 7.8

$$J_t\psi = \sum_{n=0}^{\infty} \frac{i\rho^n}{n!} M_t^n \psi = \sum_{n=0}^{\infty} \frac{i\rho^n}{n!} \widetilde{M_t^n} \psi = \sum_{n=0}^{\infty} \frac{i\rho^n}{n!} (\tilde{M}_t)^n \psi = \sum_{n=0}^{\infty} \frac{i\rho^n}{n!} \overline{M_t}^n \psi = e^{i\rho\overline{M_t}}\psi.$$

It follows from Theorem 2.2 (iii) that J_t is the unique cmx representation of $e^{i\rho\overline{M}_t}$ ∎

Proposition 7.11 *Let* $p,q,r \in \{1,2,\infty\}$ *with* $1/p + 1/q = 1/r$ *and let* $0 \prec \Phi(u)$ *and* $0 \prec \Psi(u)$ *be continuous scalar matrix valued functions such that the product matrix* $\Phi(u)\Psi(u)$ *exists for* $u \in [0,1]$.

Let $u \mapsto S_n(u)$ *and* $u \mapsto T_n(u)$ *be sequences of continuous functions* $S_n : [0,1] \to \mathcal{L}^{p}_{\rm cmx}(\mathfrak{H})_{\rm so}$ *and* $T_n : [0,1] \to \mathcal{L}^{q}_{\rm cmx}(\mathfrak{H})_{\rm so}$ *respectively such that, for each* $u \in [0,1]$,

(a) *$S_n(u)$ converges to $S(u)$ in $\mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\boldsymbol{\mathcal{V}}(S_n) \prec \Phi(u)$ $n = 1, 2, \ldots$;*

(b) *$T_n(u)$ converges to $T(u)$ in $\mathcal{L}^q_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\boldsymbol{\mathcal{V}}(T_n) \prec \Psi(u)$ $n = 1, 2, \ldots$.*

Then $S_n(u)T_n(u)$ and $S(u)T(u)$ both exist and $S_n(u)T_n(u)$ converges to $S(u)T(u)$ in $\mathcal{L}^r_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ for each $u \in [0,1]$. Moreover and $u \mapsto S(u)T(u)$ is entrywise Bochner–Lebesgue integrable and

$$\lim_{n\to\infty} \int_0^1 S_n(u)T_n(u)\, du = \int_0^1 S(u)T(u)\, du.$$

Proof. It follows from Lemma 7.7 that $S_n(u)T_n(u)$ and $S(u)T(u)$ both exist for all $u \in [0,1]$.

Suppose Φ_n is a sequence of scalar matrices with $0 \prec \Phi_n \prec \Phi$ such that $\lim_{n\to\infty} \Phi_n$is the zero matrix. Then $(\Phi_n)^i_k \Psi^k_j \leq (\Phi_n)^i_k \Psi^k_j$ for each k for all n. Now $\sum_{k=0}^{\infty} (\Phi_n)^i_k \Psi^k_j = (\Phi\Psi)^i_j < \infty$ and by a series version of the dominated convergence theorem

$$\lim_{n\to\infty} (\Phi_n\Psi)^i_j = \lim_{n\to\infty} \sum_{k=0}^{\infty} (\Phi_n)^i_k \Psi^k_j = 0$$

for all i, j. Since $\boldsymbol{\mathcal{V}}(S(u) - S_n(u)) \prec 2\Phi(u)$ it follows that $\lim_{n\to\infty} \boldsymbol{\mathcal{V}}(S(u) - S_n(u))\boldsymbol{\mathcal{V}}(T(u)) = 0$. Similarly, since $\boldsymbol{\mathcal{V}}(S_n(u))\boldsymbol{\mathcal{V}}(T(u) - T_n(u)) \prec \Phi\boldsymbol{\mathcal{V}}(T(u) - T_n(u))$ it follows that $\lim_{n\to\infty}(S_n(u))\boldsymbol{\mathcal{V}}(T(u) - T_n(u)) = 0$. Since

$$\boldsymbol{\mathcal{V}}(S(u)T(u) - S_n(u)T_n(u)) \prec \boldsymbol{\mathcal{V}}(S(u) - S_n(u))\boldsymbol{\mathcal{V}}(T(u)) + \boldsymbol{\mathcal{V}}(S_n(u))\boldsymbol{\mathcal{V}}(T(u) - T_n(u))$$

it follows that $\lim_{n\to\infty} \boldsymbol{\mathcal{V}}(S(u)T(u) - S_n(u)T_n(u)) = 0$. By Lemma 7.6 $S_n(u)T_n(u)$ converges to $S(u)T(u)$ in $\mathcal{L}^r_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ for each $u \in [0,1]$.

Since $S_n(u)T_n(u) \prec \Phi(u)\Psi(u)$ for all n and $u \in [0,1]$ it follows from the bounded convergence theorem that $u \mapsto S(u)T(u)$ is entrywise Bochner–Lebesgue integrable and

$$\lim_{n\to\infty} \int_0^1 S_n(u)T_n(u)\, du = \int_0^1 S(u)T(u)\, du. \blacksquare$$

8 Hudson–Parthasarathy Processes

Theorem 6.8 exhibits the natural correspondence between cmx semimartingales and quantum semimartingales: processes with regular integrands.

In the Hudson–Parthasarathy theory there are interesting quantum stochastic integrals whose integrands are processes of unbounded operators. Polynomials in the Brownian semimartingale $(A_A^\dagger)$, for example, do not satisfy the conditions of Theorem 6.8. A polynomial $p(\Lambda_t, A_t, A_t^\dagger)$, of degree greater than one, whose coefficients are continuously differentiable functions of t may be written as a quantum stochastic integral whose integrands are processes of unbounded operators. For example

$$A_t^2 = 2\int_0^t A_s \, dA_s.$$

In this section we show that the correspondence in Theorem 6.8 may be extended to such processes.

The integrands in the Hudson–Parthasarathy theory are adapted processes of operators with common domain $\mathcal{E}_S$ where S is an admissible subspace of $L^\infty[0,1]$ [11, §3], [20, §24]. The restriction that $S \subset L^\infty[0,1]$ imposed in [11] is unnecessary: *cf.* [20, §25] and [16, VI. 6].

Measure theoretic considerations require a slight modification of some definitions in [11], as in [20, §25]. A process $\hat{X} = \{\hat{X}_t : t \in [0,1]\}$ is *measurable* with respect to S if $t \mapsto \hat{X}_t e(g)$ is Borel measurable, for each $g \in S$.

This definition requires only that $e(g) \in \mathcal{D}(X_t)$ for almost all $t \in [0,1]$.

$\hat{X}$ is *adapted* with respect to S if, for each $t \in [0,1]$, there exists a linear transformation $\hat{X}_{t]}$ in $\mathfrak{H}_{t]}$ such that $\hat{X}_t e(g) = \mathfrak{U}_t[(\hat{X}_t e(g_{t]})) \otimes e(g_{(t})]$ for almost all $t \in [0,1]$.

For the rest of this article, unless otherwise stated, the integrands of quantum stochastic integrals and cmx quantum stochastic integrals will be adapted.

Let $p = 1, 2$ or ∞ and define the following vector spaces of operator processes in $\mathfrak{H}$

$\mathfrak{M}(\mathcal{E}_S) = \{\hat{X} : \hat{X} \text{ is measurable with respect to } S\}$;

$\mathfrak{L}^p(\mathcal{E}_S) = \{\hat{X} \in \mathfrak{M}(\mathcal{E}_S) : t \mapsto \hat{X}_t e(g) \text{ is in } L^p([0,1], \mathfrak{H}) \text{ for each } g \in S\}$;

For simplicity we let $S = L^2[0,1]$ and put $\mathcal{E} = \mathcal{E}_S$.

The usual measure theoretic conventions apply. If $\hat{X} \in \mathfrak{L}^p(\mathcal{E})$ and $g \in L^2[0,1]$ then $\hat{X}_s e(g)$ need only be defined almost everywhere on $[0,1]$. If $\hat{Y} \in \mathfrak{L}^p(\mathcal{E})$ and $\hat{Y}_t e(g) = \hat{X}_t e(g)$ for almost all t, whenever $g \in S$, then the processes $\hat{Y}$ and $\hat{X}$ are equivalent, and will normally be identified. If $g \in L^2[0,1]$ define

$$\check{X}_t e(g) = \begin{cases} \hat{X}_t e(g) & \text{if } e(g) \in \mathcal{D}(\hat{X}_t) \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, t \in [0,1].$$

Then $\check{X}$ is equivalent to $\hat{X}$ and $\mathcal{D}(\check{X}_t) = \mathcal{E}$ for all $t \in [0,t]$. If $\hat{X}$ is an adapted process in $\mathfrak{L}^p(\mathcal{E})$ then $\check{X}$ is a process of the type considered by Hudson and Parthasarathy [11, §3].

If $\hat{E} \in \mathfrak{L}^\infty(\mathcal{E})$, $\hat{F}, \hat{G} \in \mathfrak{L}^2(\mathcal{E})$ and $\hat{H} \in \mathfrak{L}^1(\mathcal{E})$ are all adapted it follows from [11, Theorem 4.4] that there is a unique process $\check{M} = \check{M}(\check{E}, \check{F}, \check{G}, \check{H})$ with $\mathcal{D}(\check{M}_t) = \mathcal{E}$ for all $t \in [0,1]$:

$$\check{M}_t = \int_0^t \check{E}\, d\Lambda + \check{F}\, dA + \check{G}\, dA^\dagger + \check{H}\, ds$$

such that

$$\begin{aligned} \left\langle \check{M}_t e(f), e(g) \right\rangle &= \int_0^t \left\langle (f\overline{g}\check{E} + f\check{F} + \overline{g}\check{G} + \check{H})(s) e(f), e(g) \right\rangle ds \\ &= \int_0^t \left\langle (f\overline{g}\hat{E} + f\hat{F} + \overline{g}\hat{G} + \hat{H})(s) e(f), e(g) \right\rangle ds \qquad (69) \end{aligned}$$

for all $f, g \in L^2[0,1]$. The *quantum stochastic integral* $\hat{M} = \{\hat{M}_t : t \in [0,1]\}$ of the quadruple $(\hat{E}, \hat{F}, \hat{G}, \hat{H})$ is defined to be the process $\check{M} = \{\check{M}_t : t \in [0,1]\}$. It follows from (69) that this definition is consistent with that of Hudson and Parthasarathy [11, Theorem 4.4] and depends only on the equivalence classes of $\hat{E}$, $\hat{F}$, $\hat{G}$ and $\hat{H}$.

A cmx process $X \in \mathcal{L}^p_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ is a *cmx representation* of $\hat{X} \in \mathfrak{L}^p(\mathcal{E})$ if for each $g \in L^2[0,1]$ there exists a null set $\mathcal{N}_g$ such that $e(g) \in \mathfrak{D}(X_t)$ and $X_t e(g)$ is the chaos representation of $\hat{X}_t e(g)$ whenever $t \in [0,1] \setminus \mathcal{N}_g$.

We give a sufficient condition that this be the case.

The *exponential domain*, ℓ^2_{exp}, in ℓ^2 is the linear span of vectors of the form

$$\varepsilon(x) = (1, x, x^2/(2!)^{1/2}, \ldots, x^j/(j!)^{1/2}, \ldots), \quad x \in \mathbb{C}.$$

Theorem 8.1 *Let $p = 1, 2$ or ∞. If $X \in \mathcal{L}^p_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ and $\ell^2_{\text{exp}} \subset \mathcal{D}(\boldsymbol{\mathcal{V}}(T))$ then*

(i) $\tilde{X} \in \mathfrak{L}^p(\mathcal{E})$;

(ii) *If X is adapted so is $\tilde{X} \in L^p(\mathcal{E})$.*

Proof. (i) $p = \infty$, $\boldsymbol{\mathcal{V}}^i_j = \left\|X^i_j\right\|_\infty$.

$$\sum_{j=0}^{\infty} \left\|X_s{}^i_j e^j(g)\right\| \le \sum_{j=0}^{\infty} \left\|X_s{}^i_j\right\|_\infty \left\|e^j(g)\right\| \le \sum_{j=0}^{\infty} \boldsymbol{\mathcal{V}}^i_j \left\|e^j(g)\right\| = (\boldsymbol{\mathcal{V}}\varepsilon(\|g\|))^i,$$

and $\sum X_s{}^i_j e^i(g)$ is absolutely convergent in $\mathfrak{H}^i$ to $\eta^i_s(g)$ with $\|\eta^i_s\| \le \|(\boldsymbol{\mathcal{V}}\varepsilon(\|g\|))^i\|$ for almost all $s \in [0,1]$. Therefore

$$\sum_{i=0}^{\infty} \left\|\eta^i_s(g)\right\|^2 \le \sum_{i=0}^{\infty} \left\|(\boldsymbol{\mathcal{V}}\varepsilon(\|g\|))^i\right\|^2 = \|\boldsymbol{\mathcal{V}}\varepsilon(\|g\|)\|^2,$$

and $\sum_{i=0}^{\infty} \eta^i_s(g)$ is convergent, for almost all s, to $\eta_s(g)$ in $\mathfrak{H}$. For each $s \in [0,1]$ and $g \in L^2[0,1]$ define

$$\hat{X}_s e(g) = \begin{cases} \eta_s(g) & \text{whenever } \sum_{i=0}^{\infty} \eta^i_s(g) \text{ is convergent} \\ 0 & \text{otherwise.} \end{cases}$$

Since the exponential vectors are linearly independent $\hat{X}_s e(g)$ is well defined for each $g \in L^2[0,1]$ and $\hat{X}_s$ extends uniquely to a linear operator $\hat{X}_s : \mathcal{E} \to \mathfrak{H}$. Since $t \mapsto X_t{}^i_j e^i(g)$ is measurable it is clear from above that $t \mapsto \hat{X}_t e(g)$ is also measurable for each $g \in L^2[0,1]$. Moreover $\left\|\hat{X}_s e(g)\right\| \le \|\boldsymbol{\mathcal{V}}\epsilon(\|g\|)\|$ for almost all s in $[0,1]$ and $\hat{X} \in \mathfrak{L}^\infty(\mathcal{E})$.

From the definition of $\tilde{X}_t$ in Section 2 it follows that $\hat{X} = \tilde{X}$ and $\tilde{X} \in \mathfrak{L}^\infty(\mathcal{E})$.

(ii) $p=2$, $\boldsymbol{\nu}^i_j = \left\|X^i_j\right\|_2$. If $g \in L^2[0,1]$ then

$$
\begin{aligned}
\|\boldsymbol{\nu}\varepsilon(\|g\|)\|^2 &\geq \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\left\|X^i_j\right\|_2\left\|e^j(g)\right\|\right)^2 \\
&= \sum_{i,j,k=0}^{\infty}\left\|X^i_j\right\|_2\left\|X^i_k\right\|_2\left\|e^j(g)\right\|\left\|e^k(g)\right\| \\
&\geq \sum_{i,j,k=0}^{\infty}\int_0^t\left\|X_s{}^i_j\right\|\left\|X_s{}^i_k\right\|\,ds\left\|e^j(g)\right\|\left\|e^k(g)\right\| \\
&= \sum_{i=0}^{\infty}\int_0^t\sum_{j,k=0}^{\infty}\left\|X_s{}^i_j\right\|\left\|X_s{}^i_k\right\|\,ds\left\|e^j(g)\right\|\left\|e^k(g)\right\| \qquad (70) \\
&\geq \sum_{i=0}^{\infty}\int_0^t\left|\left\langle\sum_{j=0}^{\infty}X_s{}^i_j e^j(g),\sum_{k=0}^{\infty}X_s{}^i_k e^k(g)\right\rangle\right|\,ds \\
&\geq \sum_{i=0}^{\infty}\int_0^t\left\|\sum_{j=0}^{\infty}X_s{}^i_j e^j(g)\right\|^2\,ds \\
&= \int_0^t\sum_{i=0}^{\infty}\left\|\sum_{j=0}^{\infty}X_s{}^i_j e^j(g)\right\|^2\,ds.
\end{aligned}
$$

It follows from the Lebesgue series theorem that, for almost all $s \in [0,1]$, the sum $\sum_{j=0}^{\infty} X_s{}^i_j e^j(g)$ is absolutely convergent in $\mathfrak{H}^j$ to η^j_s and the direct sum $\bigoplus_{i=0}^{\infty}\eta^j_s$ is convergent in $\mathfrak{H}$ to η_s. Furthermore the above inequalities show that $s \mapsto \|\eta_s\|$ is in $L^2[0,1]$. For each $s \in [0,1]$ and $g \in L^2[0,1]$ define

$$
\hat{X}_s e(g) = \begin{cases} \eta_s(g) & \text{whenever } \sum_{i=0}^{\infty}\eta^i_s(g) \text{ is convergent} \\ 0 & \text{otherwise.} \end{cases}
$$

As in (i) $\hat{X}_s$ extends uniquely to a linear operator $\hat{X}_s : \mathcal{E} \to \mathfrak{H}$ such that $s \mapsto \hat{X}e(g)$ is measurable for all $g \in L^2[0,1]$.

For almost all s

$$
\left\|\hat{X}_s e(g)\right\|^2 = \sum_{i=0}^{\infty}\left\|\sum_{j=0}^{\infty}X_s{}^i_j e^j(g)\right\|^2.
$$

Since $\mathcal{V}$ is defined on the exponential domain, it follows from the inequalities (70) that

$$\int_0^t \left\| \hat{X}_s e(g) \right\|^2 ds = \int_0^t \sum_{i=0}^{\infty} \left\| \sum_{j=0}^{\infty} X_s{}^i_j e^j(g) \right\|^2 ds \leq \|\mathcal{V}\varepsilon(\|g\|)\|^2 < \infty$$

and $\hat{X}$ belongs to $\mathfrak{L}^2(\mathcal{E})$. As in (i) $\hat{X} = \tilde{X}$ and $\tilde{X} \in \mathfrak{L}^2(\mathcal{E})$.

(iii) $p = 1$, $\mathcal{V}^i_j = \left\| X^i_j \right\|_1$. If $g \in L^2[0,1]$ then

$$\sum_{j=0}^{\infty} \int_0^{\tau} \left\| X_s{}^i_j e^j(g) \right\| ds \leq \sum_{j=0}^{\infty} \mathcal{V}^i_j \left\| e^j(g) \right\| < \infty \tag{71}$$

and $\sum_{j=0}^{\infty} X_s{}^i_j e^j(g)$ is almost everywhere absolutely convergent to $\eta^i_s(g)$ in $\mathfrak{H}^i$. If $\psi \in \mathfrak{H}$ then

$$\sum_{i=0}^{\infty} \left\| \varphi^i \right\| \int_0^1 \left\| \eta^i_s(g) \right\| ds \leq \sum_{i,j=0}^{\infty} \left\| \varphi^i \right\| \mathcal{V}^i_j \left\| e^j(g) \right\| = \sum_{i=0}^{\infty} \left\| \varphi^i \right\| \sum_{i=0}^{\infty} \mathcal{V}^i_j \left\| e^j(g) \right\| < \infty$$

since $(\|\varphi^0\|, \|\varphi^1\|, \|\varphi^2\|, \ldots)^t \in \ell^2$. Therefore, for each $x \in \ell^2$, the sum $\sum_{i=0}^{\infty} x^i \int_0^1 \|\eta^i_s(g)\| \, ds$ is convergent. It follows from the uniform boundedness theorem and Lebesgue's dominated convergence theorem that

$$\sum_{i=0}^{\infty} \int_0^1 \left\| \eta^i_s(g) \right\| ds < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \left\| \eta^i_s(g) \right\| < \infty$$

almost everywhere. Therefore $\sum_{i=0}^{\infty} \|\eta^i_s(g)\|^2 < \infty$ almost everywhere. For each $s \in [0,1]$ and $g \in L^2[0,1]$ define

$$\hat{X}_s e(g) = \begin{cases} \eta_s(g) & \text{whenever } \sum_{i=0}^{\infty} \eta^i_s(g) \text{ is convergent} \\ 0 & \text{otherwise.} \end{cases}$$

As in (ii) this extends uniquely to a linear operator $\hat{X}_s : \mathcal{E} \to \mathfrak{H}$.

It follows from (71) that

$$\int_0^1 \left\| \tilde{X}_s e(g) \right\| ds = \int_0^1 \left\| \bigoplus_{i=0}^{\infty} \sum_{j=0}^{\infty} X_s{}^i_j e^j(g) \right\| ds \leq \int_0^1 \sum_{i=0}^{\infty} \left\| \sum_{j=0}^{\infty} X_s{}^i_j e^j(g) \right\| ds < \infty$$

so that the process $\tilde{X}$ is in $\mathfrak{L}^1(\mathcal{E})$. From the definition of $\tilde{X}_t$ in Section 2 it follows that $\hat{X} = \tilde{X}$ and $\tilde{X} \in \mathfrak{L}^1(\mathcal{E})$.

(ii) This follows from Theorem 4.2 (ii). ∎

Theorem 8.2 *Let (E, F, G, H) be an adapted integrable cmx quadruple such that $\ell^2_{\exp} \subset \mathcal{D}(\boldsymbol{\mathcal{V}})$ whenever $\boldsymbol{\mathcal{V}}$ is one of the scalar matrices $\boldsymbol{\mathcal{V}}(E)$, $\boldsymbol{\mathcal{V}}(F)$, $\boldsymbol{\mathcal{V}}(G)$, $\boldsymbol{\mathcal{V}}(H)$.*

If $0 \le t \le 1$ the quantum stochastic integral

$$\hat{M}_t = \int_0^t (\tilde{E}\, d\Lambda + \tilde{F}\, dA + \tilde{G}\, dA^\dagger + \tilde{H}\, ds) \tag{72}$$

exists with $\mathcal{D}(\hat{M}_t) = \mathcal{E}$. Moreover, if

$$M_t = \int_0^t E_s\, d\Lambda_s + F_s\, dA_s + G_s\, dA_s^\dagger + H_s\, ds$$

then $\mathcal{E} \subset \mathfrak{D}(M_t)$ and M is a cmx representation of $\hat{M}$.

Proof. Let $S = L^2[0,1]$. From the remarks preceding Theorem 8.1 and its Corollary it follows that that the quantum stochastic integral (72) exists and is the unique process such that

$$\left\langle \hat{M}_t e(f), e(g) \right\rangle \;=\; \int_0^t \left\langle (f\overline{g}\tilde{E} + f\tilde{F} + \overline{g}\tilde{G} + \tilde{H})(s) e(f), e(g) \right\rangle ds \tag{73}$$

for all $f, g \in L^2[0,1]$.

We treat separately the individual constituents of the quantum stochastic integrals.

(i) Let $z \in \mathbb{C}$, $g, h \in L^2[0,1]$ and let $M_t = M_t(E : d\Lambda)$. If $\varphi \in \mathfrak{H}^i$ then

$$\begin{aligned}
\left|\left\langle M_t{}^i_j e^j(g), \varphi \right\rangle\right| &= \left| \int_0^t \left\langle g(s) E_s{}^{i-1}_{j-1} e^{j-1}(g), \nabla_s \varphi \right\rangle ds \right| \\
&\le \int_0^t |g(s)| \left\| E_s{}^{i-1}_{j-1} \right\| \frac{\|g\|^{j-1}}{((j-1)!)^{\frac12}} \|\nabla_s \varphi\|\, ds \\
&\le \boldsymbol{\mathcal{V}}^{i-1}_{j-1} \frac{\|g\|^{j-1}}{((j-1)!)^{\frac12}} \int_0^t |g(s)|\, \|\nabla_s \varphi\|\, ds \\
&= \boldsymbol{\mathcal{V}}^{i-1}_{j-1} \frac{\|g\|^{j-1}}{((j-1)!)^{\frac12}} i^{\frac12} \int_0^t |g(s)| \left(\int_{[0,1]^{i-1}} |\varphi(x;s)|^2\, dx \right)^{\frac12} ds \\
&= \boldsymbol{\mathcal{V}}^{i-1}_{j-1} \frac{\|g\|^{j-1}}{((j-1)!)^{\frac12}} i^{\frac12} \|g\|\, \|\varphi\|\,.
\end{aligned}$$

Therefore $\left\| M_t{}^i_j e^j(g) \right\| \leq i^{\frac{1}{2}} \left\| g \right\| \boldsymbol{\nu}^{i-1}_{j-1} \dfrac{\left\| g \right\|^{j-1}}{((j-1)!)^{\frac{1}{2}}}$ and, since $\varepsilon(\|g\|) \in \mathfrak{D}(\boldsymbol{\nu})$, it follows that $\sum_{j=0}^{\infty} M_t{}^i_j e^j(g)$ is absolutely convergent to η^i_t in $\mathfrak{H}^i$. If $z \in \mathbb{C}$ then

$$\begin{aligned}
\left\langle \hat{M}_t e(g), e(zh) \right\rangle &= \int_0^t \left\langle g(s)\tilde{E}_s e(g), zh(s)e(zh) \right\rangle ds \\
&= \int_0^t \left\langle g(s) \sum_{j=0}^{\infty} E_s{}^i_j e^j(g), zh(s) \sum_{i=0}^{\infty} e^i(zh) \right\rangle ds \\
&= \sum_{i,j=0}^{\infty} \int_0^t \left\langle g(s) E_s{}^i_j e^j(g), zh(s) e^i(zh) \right\rangle ds \\
&= \sum_{i,j=0}^{\infty} \overline{z}^{i+1} \int_0^t \left\langle g(s) E_s{}^i_j e^j(g), h(s) e^i(h) \right\rangle ds \\
&= \sum_{i,j=1}^{\infty} \overline{z}^{i} \int_0^t \left\langle g(s) E_s{}^{i-1}_{j-1} e^{j-1}(g), h(s) e^{i-1}(h) \right\rangle ds \\
&= \sum_{i,j=1}^{\infty} \overline{z}^{i} \left\langle M_t{}^i_j e^j(g), e^i(h) \right\rangle \\
&= \sum_{i=1}^{\infty} \overline{z}^{i} \left\langle \eta^i_t, e^i(h) \right\rangle .
\end{aligned}$$

The interchange of sum and integral in the third inequality follows from the Lebesgue series theorem since

$$\sum_{j=0}^{\infty} \left\| E_s{}^i_j e^j(g) \right\| \leq \boldsymbol{\nu}^i_j \varepsilon(\|g\|) < \infty$$

for almost all $s \in [0,1]$. Now

$$\left\langle \hat{M}_t e(g), e(zh) \right\rangle = \sum_{i=0}^{\infty} \overline{z}^i \left\langle \left(\hat{M}_t e(g) \right)^i, e^i(h) \right\rangle$$

for all $z \in \mathbb{C}$ so that $\langle \eta^i, e^i(h) \rangle = \left\langle (\hat{M}_t e(g))^i, e^i(h) \right\rangle$ whenever $h \in L^2[0,1]$. Therefore $(\hat{M}_t e(g))^i = \eta^i_t$ and

$$\eta_t = (\eta^0_t, \eta^1_t, \ldots, \eta^i_t, \ldots)^t$$

belongs to $\mathfrak{H}_{t]}$ and $e(g) \in \mathfrak{D}(M_t)$ with $M_t e(g) = \eta_t = \hat{M}_t e(g)$. Thus M_t is a cmx representation of $\hat{M}_t$ for all t in $[0,1]$.

(ii) Let $z \in \mathbb{C}$, $g, h \in L^2[0,1]$ and let $M_t = M_t(F : dA)$. If $\varphi \in \mathfrak{H}^i$ then

$$\begin{aligned}
\left|\left\langle M_t{}^i_j e^j(g), \varphi\right\rangle\right| &= \left|\int_0^t \left\langle F_s{}^i_{j-1} \nabla_s e^j(g), \varphi\right\rangle\, ds\right| \\
&= \left|\int_0^t \left\langle g(s) F_s{}^i_{j-1} e^{j-1}(g), \varphi\right\rangle\, ds\right| \\
&\leq \int_0^t |g(s)| \left\|F_s{}^i_{j-1}\right\| \left\|e^{j-1}(g)\right\| \|\varphi\|\, ds \\
&\leq \|g\| \left\|F^i_{j-1}\right\|_2 \left\|e^{j-1}(g)\right\| \|\varphi\| \\
&= \boldsymbol{\nu}^i_{j-1} \left\|e^{j-1}(g)\right\| \|g\|\ \|\varphi\|\,,
\end{aligned}$$

and $\left\|M_t{}^i_j e^j(g)\right\| \leq \|g\|\, \boldsymbol{\nu}^i_j\, \|e^{j-1}(g)\|$. Since $\varepsilon(\|g\|)$ belongs to $\mathcal{D}(\boldsymbol{\nu})$ it follows that $\sum_{j=0}^\infty M_t{}^i_j e^j(g)$ is absolutely convergent to η^i_t in $\mathfrak{H}^i$. If $z \in \mathbb{C}$ then

$$\begin{aligned}
\left\langle \hat{M}_t e(g), e(zh)\right\rangle &= \int_0^t \langle g(s) F_s e(g), e(zh)\rangle\, ds \\
&= \int_0^t \left\langle g(s) \sum_{j=0}^\infty F_s{}^i_j e^j(g), \sum_{i=0}^\infty z^i e^i(h)\right\rangle ds \\
&= \sum_{i,j=0}^\infty \overline{z}^i \int_0^t \left\langle g(s) F_s{}^i_j e^j(g), e^i(h)\right\rangle ds \\
&= \sum_{i,j=0}^\infty \overline{z}^i (j+1)^{\frac{1}{2}} \int_0^t \left\langle F_s{}^i_j \nabla_s e^{j+1}(g), e^i(h)\right\rangle ds \\
&= \sum_{i,j=0}^\infty \overline{z}^i \left\langle M_t{}^i_{j+1} e^{j+1}(g), e^i(h)\right\rangle \\
&= \sum_{i=0}^\infty \overline{z}^i \left\langle \eta^i_t, e^i(h)\right\rangle.
\end{aligned}$$

Since $\left\langle \hat{M}_t e(g), e(zh)\right\rangle = \sum_{i=0}^\infty \overline{z}^i \left\langle (\hat{M}_t e(g))^i, e^i(h)\right\rangle$ it follows as in (i) that $\eta^i_t = (\hat{M}_t e(g))^i$ for all i and that M_t is a cmx representation of $\hat{M}_t$ for all $t \in [0,1]$).

(iii) Let $M_t = M_t(\,dA^\dagger : G)$. Arguing as in (ii), if $\varphi \in \mathfrak{H}^i$ then

$$|\left\langle M_t{}^i_j e^j(g), \varphi \right\rangle| \leq i^{\frac{1}{2}} \left\| G^i_j \right\|_2 \left\| e^j(g) \right\| \, \|\varphi\| \,,$$

and M_t is a cmx representation of $\hat{M}_t$ for all $t \in [0,1]$.

(iv) The case $M_t = M_t(H : ds)$ may be dealt with similarly. ∎

Let k, m and n be non-negative integers and consider the cmx process $X_t = X_t(k,m,n)$ where $X_t(k,m,n)$ equals (the cmx representation of) $\Lambda_t^k A_t^m (A_t^\dagger)^n$. A simple matrix calculation shows that $X_t{}^i_j = 0$ unless $i = j + n - m$.

Let $\mathcal{P}$ be the space of polynomials P in $(\Lambda_t, A_t, A_t^\dagger)$ with continuously differentiable coefficients. Each polynomial P has a unique representation

$$P_t = \sum_{k,m,n=0}^{K} c_{k,m,n}(t) X_t(k,m,n).$$

If (E,F,G,H) is a quadruple in $\mathcal{P}$ then (E,F,G,H) satisfies the conditions of Theorem 8.2.

For simplicity we consider only $X_t(k,m,n)$ where $k = 1$. If $\psi^j \in \mathfrak{H}^j$ and $t \in [0,1]$ then

$$\begin{aligned}
\left\| (A_t^\dagger)^n \psi^j \right\| &\leq ((j{+}1)\ldots(j{+}n))^{\frac{1}{2}} \left\| \psi^j \right\| \\
\left\| A_t^m (A_t^\dagger)^n \psi^j \right\| &\leq ((j{+}1)\ldots(j{+}n)(j{+}n{-}1)\cdots(j{+}n{-}m))^{\frac{1}{2}} \left\| \psi^j \right\| \\
\left\| \Lambda_t A_t^m (A_t^\dagger)^n \psi^j \right\| &\leq ((j{+}1)\ldots(j{+}n)(j{+}n{-}1)\cdots(j{+}n{-}m))^{\frac{1}{2}} (j{+}n{-}m)^k \left\| \psi^j \right\| .
\end{aligned}$$

and

$$c_j = \left\| X_j^{j+n-m} \right\| \leq ((j{+}1)\ldots(j{+}n)(j{+}n{-}1)\cdots(j{+}n{-}m))^{1/2} (j{+}n{-}m)^k.$$

It follows that X is in $\mathcal{L}^p_{\rm cmx}(\mathfrak{H})_{\rm so}$ with $\boldsymbol{\nu}^i_j = 0$ if $i \neq j+n-m$ and $\boldsymbol{\nu}^{j+n-m}_j \leq 2c_j$ for $p = 1, 2, \infty$. By the limit ratio test

$$\sum_{j=0}^{\infty} c_j^2 \frac{x^j}{j!} < \infty$$

for all $x \in \mathbb{R}$ and the domain of $\boldsymbol{\nu}(X)$ contains $\ell^2_{\rm exp}$.

Since $\tilde{X}_t(k,m,n)$ equals $\Lambda_t^k A_t^m (A_t^\dagger)^n$ on $\mathcal{E}$ we have the following corollary.

Corollary 8.3 *Suppose the quadruple (E,F,G,H) consists of polynomials in the basic cmx processes $\Lambda, A, A^\dagger, t$. Then the quadruple $(\tilde{E},\tilde{F},\tilde{G},\tilde{H})$ consists of the corresponding polynomials in the basic operator processes all defined on $\mathcal{E}$. Let M be the cmx quantum stochastic integral*

$$M_t = \int_0^t E_s\, d\Lambda_s + F_s\, dA_s + G_s\, dA_s^\dagger + H_s\, ds.$$

Then $\mathcal{E} \subset \mathfrak{D}(M_t)$ for each $t \in [0,1]$ and, for each $g \in L^2[0,1]$,

$$\tilde{M}_t e(g) = \int_0^t (\tilde{E}_s\, d\Lambda_s + \tilde{F}_s\, dA_s + \tilde{G}_s\, dA_s^\dagger + \tilde{H}_s\, ds) e(g).$$

9 The Ito Product Formula

Our original proof of the Local Ito Formula was in the spirit of [11] and used approximation by step processes. It was excessively long and we thank Mr A Belton for his very concise proof.

He uses the Evans notation [7]. $A_t{}^1_1 = \Lambda_t$, $A_t{}^0_1 = A_t$, $A_t{}^1_0 = A_t^\dagger$, $A_t{}^0_0 = tI$. If (E,F,G,H) is an adapted integrable cmx quadruple put

$$M_t{}^1_1(E) = \int_0^t E_s\, dA_s{}^1_1, \qquad M_t{}^0_1(F) = \int_0^t F_s\, dA_s{}^0_1,$$
$$M_t{}^1_0(G) = \int_0^t dA_s{}^1_0 G_s, \qquad M_t{}^0_0(H) = \int_0^t H_s\, dA_s{}^0_0,$$

We thank Mr A Belton for his assistance in proving the following lemma.

Lemma 9.1 (Local Ito Formula) *If $\alpha,\beta,\gamma,\delta \in \{0,1\}$ let $X \in \mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $Y \in \mathcal{L}^q_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ be adapted processes, where $p = 2(2-\alpha-\beta)^{-1}$ and $q = 2(2-\gamma-\delta)^{-1}$. If $i,j,\mu,\nu \in \mathbb{N}$ and $\nu+\beta = \mu+\gamma$ then*

$$\begin{aligned} M_t{}^\alpha_\beta(X)^{i+\alpha}_{\nu+\beta} M_t{}^\gamma_\delta(Y)^{\mu+\gamma}_{j+\delta} &= \int_0^t M_v{}^\alpha_\beta(X)^{i+\alpha-\gamma}_\mu Y_v{}^\mu_j\, dA_v{}^\gamma_\delta \\ + \int_0^t X_u{}^i_\nu M_u{}^\gamma_\delta(Y)^\nu_{j+\delta-\beta}\, dA_u{}^\alpha_\beta &+ \delta^\nu_\mu \delta^1_\beta \delta^\gamma_1 \int_0^t (X_s{}^i_\nu Y_s{}^\mu_j)\, dA_s{}^\alpha_\delta \end{aligned} \quad (74)$$

Proof. In the following formulae we use two exponents of ∇: $\nabla^1 = \nabla$ and $\nabla^0 = I$. If $\psi \in \mathfrak{H}^{j+\gamma}$ and $\varphi \in \mathfrak{H}^{i+\alpha}$ then

$$\begin{aligned}\left\langle M_t{}^{\gamma}_{\delta}(Y)^{\mu+\gamma}_{j+\delta}\psi, M_t{}^{\beta}_{\alpha}(X^*)^{\nu+\beta}_{i+\alpha}\varphi\right\rangle &= \\ \int_0^t\int_0^t \left\langle \nabla^\beta(u)Y^\mu_j\nabla^\delta(v)\psi, \nabla^\gamma(v)(X^*_u)^\nu_i\nabla^\alpha(u)\varphi\right\rangle dudv & \quad (75)\\ +\,\delta^\mu_\nu\delta^\gamma_1\delta^1_\beta\int_0^t \left\langle Y_s{}^\mu_j\nabla^\delta(s)\psi, (X^*_s)^\nu_i\nabla^\alpha(s)\varphi\right\rangle ds. & \quad (76)\end{aligned}$$

This identity may be checked for each quadruple $\alpha, \beta, \gamma, \delta \in \{0, 1\}$. The entries in the left-hand side are given by the Definitions (40), (41), (42) and (43). The terms $\mathcal{S}$ are then replaced by terms in ∇ using Proposition 3.1 which shows that $\nabla = \mathcal{S}^*$ and gives commutation relations for $\mathcal{S}$ and ∇.

When $\nu = \mu$ the integral in (76) is

$$\int_0^t \left\langle X_s{}^i_\nu Y_s{}^\nu_j\nabla^\delta(s), \nabla^\alpha(s)\varphi\right\rangle ds = \left\langle\left(\int_0^t X_s{}^i_\nu Y_s{}^\nu_j\, dA_s{}^\alpha_\delta\right)\psi, \varphi\right\rangle. \quad (77)$$

The double integral over $\{0 \le u,\, v \le t\}$ in (76) is the sum, $I_1 + I_2$, of the integrals over $\{0 \le u \le v \le t\}$ and $\{0 \le v < u \le t\}$. Since X is adapted it follows from Theorem 4.6 that $\nabla^\gamma(v)(X^*_u)^\nu_i\nabla^\alpha(u)\varphi = (X^*_u)^{\nu-\gamma}_{i-\gamma}\nabla^\alpha(u)\nabla^\gamma(v)\varphi$ whenever $u \le v$. Therefore

$$\begin{aligned} I_1 &= \int_0^t\int_0^v \left\langle \nabla^\beta(u)Y_v{}^\mu_j\nabla^\delta(v)\psi, (X^*_u)^{\nu-\gamma}_{i-\gamma}\nabla^\alpha(u)\nabla^\gamma(v)\varphi\right\rangle dudv \\ &= \int_0^t\int_0^v \left\langle X_u{}^{i-\gamma}_{\nu-\gamma}\nabla^\beta(u)Y_v{}^\mu_j\nabla^\delta(v)\psi, \nabla^\alpha(u)\nabla^\gamma(v)\varphi\right\rangle dudv \\ &= \int_0^t \left\langle M_v{}^\alpha_\beta(X)^{i+\alpha-\gamma}_\nu Y_v{}^\nu_j\nabla^\delta(v)\psi, \nabla^\gamma(v)\varphi\right\rangle dv \\ &= \left\langle\left(\int_0^t M_v{}^\alpha_\beta(X)^{i+\alpha-\gamma}_\nu Y_v{}^\nu_j\, dA_v{}^\gamma_\delta\right)\psi, \varphi\right\rangle \qquad (78)\end{aligned}$$

Similarly

$$I_2 = \left\langle\left(\int_0^t X_u{}^i_\nu M_u{}^\gamma_\delta(Y)^\nu_{j+\delta-\beta}\, dA_u{}^\alpha_\beta\right)\psi, \varphi\right\rangle. \quad (79)$$

The identity (74) follows immediately from (77), (78) and (79). ∎

Corollary 9.2 (Quantum Ito Product Formula) *If the product processes*

$$M^\alpha_\beta(X)Y, \qquad XM^\gamma_\delta(Y), \qquad \delta^1_\beta\delta^\gamma_1 XY$$

exist in $\mathcal{L}^q_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, $\mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *and* $\mathcal{L}^r_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, *where* $1/p+1/q=1/r$ *then the product process* $M^\alpha_\beta(X)M^\gamma_\delta(Y)$ *exists in* $\mathcal{L}^\infty_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *and*

$$\begin{aligned} M_t{}^\alpha_\beta(X)M_t{}^\gamma_\delta(Y) &= \int_0^t M_s{}^\alpha_\beta(X)Y_s\,dA_s{}^\gamma_\delta + \int_0^t X_sM_s{}^\gamma_\delta(Y)\,dA_s{}^\alpha_\beta \\ &\quad +\delta^1_\beta\delta^\gamma_1\int_0^t X_sY_s\,dA_s{}^\alpha_\delta. \end{aligned} \tag{80}$$

Proof. This follows directly from the identity (74) together with the norm bounds (47), (48) and (49) in Theorem 6.1. ∎

We give sufficient conditions that the conditions of Corollary 9.2 are satisfied.

Corollary 9.3 *Suppose* $\alpha,\beta,\gamma,\delta,p,q,X,Y$ *satisfy the conditions of Lemma* 9.1 *and define scalar matrices* $\boldsymbol{\mathcal{K}}(X),\boldsymbol{\mathcal{K}}(Y)$ *by the formulae*

$$\boldsymbol{\mathcal{K}}(X)^i_j = i^{\frac{\alpha}{2}}\boldsymbol{\mathcal{V}}(X)^{i-\alpha}_{j-\beta}j^{\frac{\beta}{2}}, \quad \boldsymbol{\mathcal{K}}(Y)^i_j = i^{\frac{\gamma}{2}}\boldsymbol{\mathcal{V}}(X)^{i-\gamma}_{j-\delta}j^{\frac{\delta}{2}}.$$

If the scalar matrix $\boldsymbol{\mathcal{K}}(X)\boldsymbol{\mathcal{V}}(Y)+\boldsymbol{\mathcal{V}}(X)\boldsymbol{\mathcal{K}}(Y)+\boldsymbol{\mathcal{V}}(X)\boldsymbol{\mathcal{V}}(Y)$ *has finite entries then* X *and* Y *satisfy the conditions of Corollary* 9.2 *and the quantum Ito product formula* (80) *is valid.*

Proof. It follows from the norm bounds (47), (48) and (49), that the r.h.s of (74) is bounded by

$$\begin{aligned} &(i+\alpha)^{\frac{\gamma}{2}}(j+\delta)^{\frac{\delta}{2}}(i+\alpha-\gamma)^{\frac{\alpha}{2}}\mu^{\frac{\beta}{2}}\left\|X^{i-\gamma}_{\mu-\beta}\right\|_p\left\|Y^\mu_\gamma\right\|_q \\ &\quad +(i+\alpha)^{\frac{\alpha}{2}}(j+\delta)^{\frac{\beta}{2}}(j+\delta-\beta)^{\frac{\delta}{2}}\nu^{\frac{\gamma}{2}}\left\|X^i_\nu\right\|_p\left\|Y^{\nu-\gamma}_{j-\beta}\right\|_q \\ &\qquad +\delta^1_\beta\delta^\gamma_1(i+\alpha)^{\frac{\alpha}{2}}(j+\delta)^{\frac{\delta}{2}}\left\|X^i_\nu\right\|p\left\|Y^\nu_j\right\|_q \\ &\le (j+\delta)^{\frac{\delta}{2}}(i+\alpha-\gamma)^{\frac{\alpha}{2}}\boldsymbol{\mathcal{K}}(X)^i_\mu\boldsymbol{\mathcal{V}}(Y)^\mu_j + (i+\alpha)^{\frac{\alpha}{2}}(j+\delta-\beta)^{\frac{\delta}{2}}\boldsymbol{\mathcal{V}}(X)^i_\nu\boldsymbol{\mathcal{K}}(Y)^\nu_j \\ &\qquad +\delta^1_\beta\delta^\gamma_1(i+\alpha)^{\frac{\alpha}{2}}(j+\delta)^{\frac{\delta}{2}}\boldsymbol{\mathcal{V}}(X)^i_\nu\boldsymbol{\mathcal{V}}(Y)^\nu_j. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{\nu=0}^\infty\left\|M_t{}^\alpha_\beta(X)^{i+\alpha}_{\nu+\beta}M_t{}^\gamma_\delta(Y)^{\mu+\gamma}_{j+\delta}\right\| &\le (j+\delta)^{\frac{\delta}{2}}(i+\alpha-\gamma)^{\frac{\alpha}{2}}(\boldsymbol{\mathcal{K}}(X)\boldsymbol{\mathcal{V}}(Y))^i_j \\ +(i+\alpha)^{\frac{\alpha}{2}}(j+\delta-\beta)^{\frac{\delta}{2}}(\boldsymbol{\mathcal{V}}(X)\boldsymbol{\mathcal{K}}(Y))^i_j &+ \delta^1_\beta\delta^\gamma_1(i+\alpha)^{\frac{\alpha}{2}}(j+\delta)^{\frac{\delta}{2}}(\boldsymbol{\mathcal{V}}(X)\boldsymbol{\mathcal{V}}(Y))^i_j. \end{aligned}$$

which is finite for all i, j. It follows that the the product process $M_t{}^{\alpha}_{\beta}(X) M_t{}^{\gamma}_{\delta}(Y)$ exists.

Now

$$\begin{aligned} \left\| M_v{}^{\alpha}_{\beta}(X)^{i+\alpha-\gamma}_{\mu} Y_v{}^{\mu}_{j} \right\| &\leq (i+\alpha-\gamma)^{\frac{\alpha}{2}} \mu^{\frac{\beta}{2}} \left\| X^{i-\gamma}_{\mu-\beta} \right\|_p \left\| Y_v{}^{\mu}_{j} \right\| \\ &\leq (i+\alpha-\gamma)^{\frac{\alpha}{2}} \mathcal{K}(X)^{i}_{\mu} \left\| Y_v{}^{\mu}_{j} \right\|, \end{aligned}$$

and

$$\sum_{\mu=0}^{\infty} \left\| M^{\alpha}_{\beta}(X)^{i+\alpha-\gamma}_{\mu} Y^{\mu}_{j} \right\|_q \leq (i+\alpha-\gamma)^{\frac{\alpha}{2}} \sum_{\mu=0}^{\infty} \mathcal{K}(X)^{i}_{\mu} \mathcal{V}(Y)^{\mu}_{j} < \infty.$$

Therefore the product chaos matrix $M_s{}^{\alpha}_{\beta}(X) Y_s$ exists for almost all $s \in [0,1]$ and defines a cmx process in $\mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$. Similar arguments show that the product chaos matrix $X_s M_s{}^{\gamma}_{\delta}(Y)$ exists almost everywhere and defines a process in $\mathcal{L}^p_{cmx}(\mathfrak{H})_{\mathrm{so}}$ and that, for $\beta = \delta = 1$, $s \mapsto X_s Y_s$ defines a process in the appropriate $\mathcal{L}^r_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$.

Taking weak sums on both sides and using the DCT yields (80). ∎

In particular if X and Y are $(2k+1)$-diagonal matrices for some $k \in \mathbb{N}$ then the conditions of Corollary 9.3 are satisfied. Since the basic processes are tri-diagonal any polynomial in them is $(2k+1)$-diagonal for some $k \in \mathbb{N}$.

Corollary 9.4 *If X and Y are polynomials in the basic processes then the quantum Ito product formula* (80) *is valid.*

10 Quantum Duhamel Formula

In this section we show that the quantum Duhamel formula [28, Proposition 5.2] for regular quantum semimartingales remains true for some quantum stochastic integrals and some essentially self-adjoint quantum semimartingales. The proofs, which use series expansions and Nelson's analytic vector theorem, follow the same lines as the proof for regular quantum semimartingales.

Theorem 10.1 *Let (E, F, F^*, H) be an adapted symmetric integrable cmx quadruple and let $\mathcal{K}$ be the control matrix of the process*

$$M_t = \int_0^t (E\, d\Lambda + F\, dA + F^*\, dA^\dagger + H\, ds). \tag{81}$$

If $\epsilon > 0$ and $\ell_{00} \subset \mathcal{A}_\epsilon(\mathcal{K})$ then, for $-\epsilon < p < \epsilon$,

(i) *the product cmx processes*

$$M^\alpha F^\mu (M+E)^\beta G^\nu M^\gamma \quad \text{exists in } \mathcal{L}^p_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}, \quad p = \frac{2}{\mu+\nu} \tag{82}$$

whenever $\alpha, \beta, \gamma \in \mathbb{N}$ and $\mu, \nu \in \{0, 1\}$.

(ii) *for $n = 1, 2, \ldots$ the formulae*

$$\begin{aligned}
E_n &= (M+E)^n - M^n \\
F_n &= \sum_{\alpha+\beta=n-1} M^\alpha F (M+E)^\beta \\
(F_n)^* &= \sum_{\alpha+\beta=n-1} (M+E)^\alpha F^* M^\beta \\
H_n &= \sum_{\alpha+\beta=n-1} M^\alpha H M^\beta + \sum_{\alpha+\beta+\gamma=n-2} M^\alpha F (M+E)^\beta F^* M^\gamma
\end{aligned}$$

define an adapted symmetric integrable cmx quadruple $(E_n, F_n, (F_n)^*, H_n)$.

(iii) *$M^n = \{(M_t)^n : t \in [0,1]\}$ is a symmetric cmx quantum stochastic integral with*

$$(M_t)^n = \int_0^t (E_n\, d\Lambda + F_n\, dA + (F_n)^*\, dA^\dagger + H_n\, ds), \quad n = 1, 2, \ldots. \tag{83}$$

(iv) *$\mathfrak{H}_{00} \subset \mathcal{A}_\epsilon(M_t) \cap \mathcal{A}_\epsilon(\tilde{M}_t)$ and $\mathfrak{H}_{00} \subset \mathcal{A}_\epsilon((M+E)_t) \cap \mathcal{A}_\epsilon(\widetilde{(M+E)}_t)$ for $t \in [0,1]$.*

(v) *$\tilde{M}$ and $\widetilde{M+E}$ are essentially self-adjoint with core $\mathfrak{H}_{00}$.*

(vi) *The series* $I + \sum_{k=1}^{\infty} (ipM)^k/k!$ *is convergent in* $\mathcal{L}^{\infty}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *to a process* $J(p)$ *and*

$$\tilde{J}_t(p) = e^{ip\overline{M}_t}$$

(vii) *The series* $I + \sum_{k=1}^{\infty} (ip(M+E))^k/k!$ *is convergent in* $\mathcal{L}^{\infty}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *to a process* $K(p)$ *and*

$$\tilde{K}_t(p) = e^{ip\overline{M+E}_t}$$

Proof. The proofs for $M+E$ are similar to the proofs for M and will be omitted.

Since $\mathcal{A}(\mathcal{K}) \neq \emptyset$ the product matrix $\mathcal{K}^k$ exists for all $k \in \mathbb{N}$ by definition. If $\mathcal{V} = \mathcal{V}(M)$ is the scalar matrix of M then $\mathcal{K} + \mathcal{V} \prec 2\mathcal{K}$ and $(\mathcal{K} + \mathcal{V})^k$ exists for all $k \in \mathbb{N}$. Since $\mathcal{K}$ and $\mathcal{V}$ have non-negative entries the product scalar matrix $\mathcal{K}^{\alpha_1}\mathcal{V}^{\beta_1} \cdots \mathcal{K}^{\alpha_k}\mathcal{V}^{\beta_k}$ exists whenever $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ are non-negative integers with $\alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_k = n$. Moreover $(\mathcal{K} + \mathcal{V})^n$ is a finite sum of such monomial products.

(i) If $1, \alpha, \beta, \gamma \leq k$ then $\mathcal{V}(M^\alpha), \mathcal{V}(F), \mathcal{V}((M+E)^\beta), \mathcal{V}(G), \mathcal{V}(M^\gamma) \prec (\mathcal{K} + \mathcal{V})^k$. Since $M, E \in \mathcal{L}^{\infty}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, $F, G \in \mathcal{L}^{2}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, and $H \in \mathcal{L}^{1}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and the product scalar matrix $(\mathcal{K} + \mathcal{V})^{5k}$ exists for all k it follows from Lemma 7.7 that $M^\alpha F^\mu (M+E)^\beta G^\nu M^\gamma$ exists whenever $\alpha, \beta, \gamma \in \mathbb{N}$ and $\mu, \nu \in \{0, 1\}$.

(ii) It follows from (i) and Corollary 9.2 that the formulae in (ii) define cmx processes E_n, F_n, $(F_n)^*$ and H_n. The following algebraic identities follow directly from those formulae.

$$\begin{aligned}
E_{n+1} &= M(M+E)^n - M^{n+1} + EM^n + E(M+E)^n - EM^n \\
&= ME_n + EM_n + EE_n; \\
F_{n+1} &= \sum_{\alpha+\beta=n-1} M^{\alpha+1} F (M+E)^\beta + FM^n + F(M+E)^n - FM^n \\
&= MF_n + FM^n + FE_n; \\
(F_{n+1})^* &= \sum_{\alpha+\beta=n-1} M(M+E)^\alpha M^\beta + F^* M^n + \sum_{\alpha+\beta=n-1} E(M+E)^\alpha F^* M^\beta \\
&= M(F_n)^* + F^* M^n + E(F_n)^*; \\
H_{n+1} &= HM^n + \sum_{\alpha+\beta=n-1} M^{\alpha+1} H M^\beta + \sum_{\alpha+\beta+\gamma=n-2} M^{\alpha+1} F (M+E)^\beta F^* M^\gamma
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\beta+\gamma=n-1} F(M+E)^{\beta}F^*M^{\gamma}\\
=\;& MH_n + HM^n + F(F_n)^*.
\end{aligned}
$$

The quadruple (E_n, F_n, G_n, H_n) is integrable by (i) and adapted by Corollary 4.7. It follows from (i) and Lemma 2.1 than $G_n = (F_n)^*$ for all $n \in \mathbb{N}$.

(iii) Applying the Quantum Ito product formula (80) to $M_t M_t^n$, where M and M^n are given by (81)and (83), gives

$$
\begin{aligned}
M_t^{n+1} &= \int_0^t (M\,dM^n + dM M^n + dM\,dM^n)\\
&= \int_0^t \big((ME_n + EM^n + EE_n)\,d\Lambda + (MF_n + FM^n + FE_n)\,dA\\
&\quad +(M(F_n)^* + F^*M^n + E(F_n)^*)dA^\dagger + (MH_n + HM^n + F(F_n)^*)ds\big)\\
&= \int_0^t (E_{n+1}d\Lambda + F_{n+1}dA + F_{n+1}^*dA^\dagger + H_{n+1}ds).
\end{aligned}
$$

(iv), (v), (vi) Since $\mathcal{V}(M) \prec \mathcal{K}$ and $\ell_{00} \subset \mathcal{A}_\epsilon(\mathcal{K})$ implies that $\ell_{00} \subset \mathcal{A}_\epsilon(\mathcal{V}(M))$. Parts (iv), (v) and (vi) then follow from Theorem 7.10. ∎

Under the above conditions we define e^{ipM_t} and $e^{ip(M+E)_t}$ to be the chaos matrices $J_t(p)$ and $K_t(p)$ respectively. We now have the following chaos matrix version of the quantum Duhamel formula.

Corollary 10.2 (Cmx Duhamel Formula) *The chaos matrix process $\{e^{ipM_t} : t \in [0,1]\}$ has a representation as a cmx quantum stochastic integral*

$$
e^{ipM_t} = I + \int_0^t (E_{\exp(ipM)}\,d\Lambda + F_{\exp(ipM)}\,dA + G_{\exp(ipM)}\,dA^\dagger + H_{\exp(ipM)}\,ds). \tag{84}
$$

The integrands form an adapted integrable quadruple and are defined by the formulae

$$
\begin{aligned}
E_{\exp(ipM)}(t) &= ip\int_0^1 e^{i(1-u)pM_t}E_t e^{iupM_t}\,du = e^{ip(M_t+E_t)} - e^{ipM_t}\\
F_{\exp(ipM)}(t) &= ip\int_0^1 e^{i(1-u)pM_t}F_t e^{iup(M_t+E_t)}\,du
\end{aligned}
$$

$$
\begin{aligned}
G_{\exp(ipM)}(t) &= ip\int_0^1 e^{i(1-u)p(M_t+E_t)}F_t^* e^{iupM_t}\,du \\
H_{\exp(ipM)}(t) &= ip\int_0^1 e^{i(1-u)pM_t}H_t e^{iupM_t}\,du \\
&\qquad -p^2\int_0^1\int_0^1 ue^{i(1-u)pM_t}F_t e^{iu(1-v)p(M_t+E_t)}F_t^* e^{iuvpM_t}\,du\,dv.
\end{aligned} \tag{85}
$$

They have power series representations

$$
E_{\exp(ipM)} = \sum_{n=0}^{\infty}\frac{(ip)^n E_n}{n!},\quad F_{\exp(ipM)} = \sum_{n=0}^{\infty}\frac{(ip)^n F_n}{n!},\quad G_{\exp(ipM)} = \sum_{n=0}^{\infty}\frac{(ip)^n F_n^*}{n!},
$$

$$
H_{\exp(ipM)} = \sum_{n=0}^{\infty}\frac{(ip)^n H_n}{n!},
$$

the series converging in $\mathcal{L}^{\infty}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, $\mathcal{L}^{2}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$, $\mathcal{L}^{2}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *and* $\mathcal{L}^{1}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ *respectively.*

Proof. There is no loss of generality in taking $\epsilon > 1$ and $p = 1$. Both M and $M + E$ satisfy the conditions of Theorem 7.10.

If $0 \le u \le 1$ define $S_k(u)$ in $\mathcal{L}^{\infty}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $T_k(u)$ in $\mathcal{L}^{2}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ by

$$
S_k(u) = \sum_{\alpha=0}^{k}\frac{(i(1-u)M)^{\alpha}}{\alpha!},\qquad T_k(u) = \sum_{\beta=0}^{k}(iF)\frac{(iu(M+E))^{\beta}}{\beta!}.
$$

By Theorem 10.1, the product processes $M^{\alpha}F(M+E)^{\beta}$ exist in $\mathcal{L}^{2}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ for all $\alpha, \beta \in \mathbb{N}$. Therefore product process $S_k(u)T_k(u)$ exists for all $u \in [0,1]$ and

$$
S_k(u)T_k(u) = \sum_{\alpha,\beta=0}^{k}\frac{(i(1-u)M)^{\alpha}}{\alpha!}(iF)\frac{(iu(M+E))^{\beta}}{\beta!}
$$

Now $u \mapsto S_k(u)$ and $u \mapsto T_k(u)$ are continuous functions into $\mathcal{L}^{\infty}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\mathcal{L}^{2}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ respectively and:

(a) $S_k(u)$ converges to $e^{i(1-u)M}$ in $\mathcal{L}^{\infty}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\mathcal{V}(S_k) \prec e^{(1-u)\mathcal{K}}$ $k = 1, 2, \ldots$;

(b) $T_k(u)$ converges to $(iF)e^{iuM}$ in $\mathcal{L}^{2}_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ and $\mathcal{V}(T_k) \prec \mathcal{K}e^{u\mathcal{K}}$ $k = 1, 2, \ldots$.

Therefore $\mathcal{V}(S_k(u))\mathcal{V}(T_k(u)) \prec \mathcal{K}e^{\mathcal{K}}$ for all $u \in [0,1]$ so by Proposition 7.11 it follows that $S_k(u)T_k(u)$ converges in $\mathcal{L}^2_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ to $e^{i(1-u)M}(iF)e^{iu(M+E)}$ for each $u \in [0,1]$ and

$$\begin{aligned}\lim_{k\to\infty}\int_0^1 S_k(u)T_k(u)\,du &= \lim_{k\to\infty}\sum_{\alpha,\beta=0}^{k}\frac{(iM)^\alpha(iF)(i(M+E))^\beta}{(\alpha+\beta+1)!}\\ &= \int_0^1 e^{i(1-u)M}(iF)e^{iu(M+E)}\,du = F_{\exp(iM)}.\end{aligned}$$

If $\mathcal{R}_k$ in $\mathcal{L}^2_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$ is defined by

$$\mathcal{R}_k \stackrel{\text{def}}{=} \left(\int_0^1 S_k(u)T_k(u)\,du\right)-\left(\sum_{n=0}^{k}\frac{i^nF_n}{n!}\right) = \sum_{0\le\alpha,\beta\le k<\alpha+\beta}\frac{(iM)^\alpha(iF)(i(M+E))^\beta}{(\alpha+\beta+1)!}$$

then $\lim_{k\to\infty}\mathcal{R}_k = 0$ in $\mathcal{L}^2_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$. This follows from Lemma 7.6 since

$$\mathcal{V}(\mathcal{R}_k) \prec \sum_{\alpha+\beta>k}\frac{\mathcal{K}^{\alpha+\beta+1}}{(\alpha+\beta+1)!} \prec \sum_{n=k}^{\infty}\frac{\mathcal{K}^n}{n!}$$

which tends to the zero matrix as $k\to\infty$. Therefore

$$F_{\exp(iM)} = \sum_{n=0}^{\infty}\frac{i^nF_n}{n!},$$

the series converging in $\mathcal{L}^2_{\mathrm{cmx}}(\mathfrak{H})_{\mathrm{so}}$.

The proof of the power series representations are similar and will be omitted.

It follows from (83) that

$$\begin{aligned}e^{ipM_t} &= I+\sum_{n=1}^{\infty}(ip)^n\int_0^t(E_n\,d\Lambda+F_n\,dA+(F_n)^*\,dA^\dagger+H_n\,ds)\\ &= I+\int_0^t(E_{\exp(ipM)}\,d\Lambda+F_{\exp(ipM)}\,dA+G_{\exp(ipM)}\,dA^\dagger+H_{\exp(ipM)}\,ds).\end{aligned}$$

The interchange of the integrals with the sums is justified by Theorem 6.6. ∎

The next problem is to deduce from (84) a quantum Duhamel formula for quantum stochastic integrals. When the integrands $(\hat{E}, \hat{F}, \hat{F}^*, \hat{H})$ are not required to be bounded we are unable to find reasonable sufficient conditions to ensure that integrands can be defined on $\mathcal{E}$ by the formulae (87). We therefore restrict attention to the case that M is a cmx semimartingale.

If the quantum Duhamel formula is valid for all real p then the functional quantum Ito formula may be proved as in [28, Theorem 6.2.] using the Fourier functional calculus

$$f(M_t) = \int_{\mathbb{R}} \hat{f}(p) e^{ipM_t}\, dp.$$

Corollary 10.2 only implies that pM satisfies the quantum Duhamel formula for $|p| < \epsilon$. The following corollary shows this is sufficient. Repeated applications of the quantum Ito product formula of Hudson and Parthasarathy are used to show that pM satisfies the quantum Duhamel formula for all real p. This is true in particular if M is one of the quantum semimartingales described in Proposition 7.4.

Corollary 10.3 *Let $\hat{M} = \{\hat{M}_t : t \in [0,1]\}$ be a symmetric quantum semimartingale*

$$\hat{M}_t = \int_0^t (\hat{E}_s\, d\Lambda_s + \hat{F}_s\, dA_s + \hat{F}^*\, dA_s^\dagger + \hat{H}_s\, ds)$$

with corresponding symmetric cmx semimartingale $M = \{M_t : t \in [0,1]\}$:

$$M_t = \int_0^t (E_s\, d\Lambda_s + F_s\, dA_s + F^*\, dA_s^\dagger + H_s\, ds).$$

If there exists $\epsilon > 0$ such that $\ell_{00} \subset \mathcal{A}_\epsilon(\boldsymbol{\mathcal{K}})$ where $\boldsymbol{\mathcal{K}}$ is the control matrix of M then

(i) *$\hat{M}$ is essentially self-adjoint with core $\mathfrak{H}_{00}$.*

(ii) *If $\overline{M}_t$ is the closure of $\hat{M}_t$ then e^{ipM} is a regular quantum semimartingale and pM satisfies the quantum Duhamel formula*

$$e^{ip\overline{M}_t} = I + \int_0^t (\hat{E}_{\exp(ipM)}\, d\Lambda + \hat{F}_{\exp(ipM)}\, dA + \hat{G}_{\exp(ipM)}\, dA^\dagger + \hat{H}_{\exp(ipM)}\, ds), \quad (86)$$

for all real p, where

$$\begin{aligned}
\hat{E}_{\exp(ipM)}(t) &= e^{ip(\overline{M}_t+\hat{E}_t)} - e^{ip\overline{M}_t} \\
\hat{F}_{\exp(ipM)}(t) &= ip\int_0^1 e^{i(1-u)p\overline{M}_t}\hat{F}_t e^{iup(\overline{M}_t+\hat{E}_t)}\,du \\
\hat{G}_{\exp(ipM)}(t) &= ip\int_0^1 e^{i(1-u)p(\overline{M}_t+\hat{E}_t)}\hat{F}_t^* e^{iup\overline{M}_t}\,du \qquad (87)\\
\hat{H}_{\exp(ipM)}(t) &= ip\int_0^1 e^{i(1-u)p\overline{M}_t}\hat{H}_t e^{iup\overline{M}_t}\,du \\
&\quad +(ip)^2\int_0^1\int_0^1 ue^{i(1-u)p\overline{M}_t}\hat{F}_t e^{iu(1-v)p(\overline{M}_t+\hat{E}_t)}\hat{F}_t^* e^{iuvp\overline{M}_t}\,du\,dv
\end{aligned}$$

Proof. There is no loss of generality in assuming $\epsilon = 1$; simply replace M by ϵM.

It follows from [23, Theorems VIII.25, VIII.21] that $u \mapsto e^{i(1-u)p\overline{M}_t}$ and $u \mapsto e^{iup(\overline{M}_t+\tilde{E}_t)}$ are strongly continuous. Therefore the function $u \mapsto e^{i(1-u)pM_t}F_t e^{iup(M_t+E_t)}$ belongs to $L^\infty_{\text{cmx}}(\mathcal{H})_{\text{so}}$ for almost all $t \in [0,1]$. Theorems 2.7 and 10.1 (vi), (vii) now imply that $F_{\exp(ipM)}(t)$ is a cmx representation of $\hat{F}_{\exp(ipM)}(t)$ for almost all t. The other integrands in (85) and (87) are similarly related. By Theorem 6.8 Equation (86) is true for $-1 \le p \le 1$.

Since $e^{i\overline{M}}$ is a unitary process it follows that $\hat{E}_{\exp(iM)}$ is in $L^\infty_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ while $\hat{F}_{\exp(iM)}$, $\hat{G}_{\exp(iM)}$ belong to $L^2_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ and $\hat{H}_{\exp(iM)}$ is in $L^1_{\text{cmx}}(\mathfrak{H})_{\text{so}}$ so that $e^{i\overline{M}}$ is a regular quantum semimartingale.

We now prove (86) is valid for $p = 2$.

By the quantum Ito product formula of Hudson and Parthasarathy, [1, Theorem 5.], [11, Theorem 4.5],

$$e^{2i\overline{M}_t} = I + \int_0^t e^{i\overline{M_s}}d\left(e^{i\overline{M_s}}\right) + d\left(e^{i\overline{M_s}}\right)e^{i\overline{M_s}} + d\left(e^{i\overline{M_s}}\right)d\left(e^{i\overline{M_s}}\right). \qquad (88)$$

The coefficient of $d\Lambda$ is $e^{i\overline{M}}\hat{E}_{\exp(i\overline{M})} + \hat{E}_{\exp(i\overline{M})}e^{i\overline{M}} + \hat{E}_{\exp(i\overline{M})}\hat{E}_{\exp(i\overline{M})}$ which equals

$$e^{i\overline{M}}e^{i(\overline{M}+\hat{E})} - e^{i2\overline{M}} + e^{i(\overline{M}+\hat{E})}e^{i\overline{M}} - e^{i2\overline{M}} + e^{i(2\overline{M}+2\hat{E})} + e^{i2\overline{M}} - e^{i\overline{M}}e^{i(\overline{M}+\hat{E})} - e^{i(\overline{M}+\hat{E})}e^{i\overline{M}}$$

$$= \hat{E}_{\exp(i2\overline{M})}$$

The coefficient of dA is

$$\begin{aligned}
&e^{i\overline{M}}\hat{F}_{\exp(i\overline{M})} + \hat{F}_{\exp(i\overline{M})}e^{i\overline{M}} + \hat{F}_{\exp(i\overline{M})}\hat{E}_{\exp(i\overline{M})} \\
&\quad = i\int_0^1 e^{i(2-u)\overline{M}}\hat{F}e^{iu(\overline{M}+\hat{E})}\,du + i\int_0^1 e^{i(1-u)\overline{M}}\hat{F}e^{iu(\overline{M}+\hat{E})}e^{i\overline{M}}\,du \\
&\qquad\qquad + i\int_0^1 e^{i(1-u)\overline{M}}\hat{F}e^{iu(\overline{M}+\hat{E})}\left(e^{iu(\overline{M}+\hat{E})} - e^{i\overline{M}}\right)\,du \\
&\quad = i\int_0^1 \left(e^{i(2-u)\overline{M}}\hat{F}e^{iu(\overline{M}+\hat{E})} + e^{i(1-u)\overline{M}}\hat{F}e^{i(1+u)(\overline{M}+\hat{E})}\right)\,du \\
&\quad = i\int_0^{\frac{1}{2}} \left(e^{i(1-v)2\overline{M}}2\hat{F}e^{iv(2\overline{M}+2\hat{E})} + e^{i(1/2-v)2\overline{M}}2\hat{F}e^{i(1/2+v)(2\overline{M}+2\hat{E})}\right)\,dv \\
&\quad = i\int_0^{\frac{1}{2}} e^{i(1-v)2\overline{M}}2\hat{F}e^{iv(2\overline{M}+2\hat{E})}\,dv + \int_{\frac{1}{2}}^1 e^{i(1-w)2\overline{M}}2\hat{F}e^{iw(2\overline{M}+2\hat{E})}\,dw \\
&\quad = i\int_0^1 e^{i(1-v)2\overline{M}}2\hat{F}e^{iv(2\overline{M}+2\hat{E})}\,dv \\
&\quad = \hat{F}_{\exp(i2\overline{M})}
\end{aligned}$$

Taking adjoints, or calculating as above, the coefficient of $dA^\dagger$ is

$$i\int_0^1 e^{i(1-u)(2\overline{M}+2\hat{E})}(2\hat{F})^* e^{iu2\overline{M}}\,du = \hat{G}_{\exp(i2\overline{M})}.$$

The contribution of $\hat{H}$ to the coefficient of dt is

$$\begin{aligned}
&i\int_0^1 \left(e^{i(2-u)\overline{M}}\hat{H}e^{iu\overline{M}} + e^{i(1-u)\overline{M}}\hat{H}e^{i(1+u)\overline{M}}\right)\,du \\
&\qquad = i\int_0^{\frac{1}{2}} \left(e^{i(1-v)2\overline{M}}2\hat{H}e^{iv2\overline{M}} + e^{i(1/2-v)2\overline{M}}2\hat{H}e^{i(1/2+v)2\overline{M}}\right)\,dv \\
&\qquad = i\int_0^{\frac{1}{2}} e^{i(1-v)2\overline{M}}2\hat{H}e^{iv2\overline{M}}\,dv + i\int_{\frac{1}{2}}^1 e^{i(1-w)2\overline{M}}2\hat{H}e^{iw2\overline{M}}\,dw \\
&\qquad = i\int_0^1 e^{i(1-u)2\overline{M}}2\hat{H}e^{iu2\overline{M}}\,du
\end{aligned}$$

We now calculate the contribution of $\hat{F}$ and $\hat{F}^*$ to the coefficient of dt:

$$\left(i\int_0^1 e^{i(1-u)\overline{M}}\hat{F}e^{iu(\overline{M}+\hat{E})}\,du\right)\left(i\int_0^1 e^{i(1-v)(\overline{M}+\hat{E})}\hat{F}^* e^{iv\overline{M}}\,dv\right)$$

$$= -\int_0^1 \int_0^1 e^{i(1-u)\overline{M}} \hat{F} e^{i(u+1-v)(\overline{M}+\hat{E})} \hat{F}^* e^{iv\overline{M}}\, dudv \quad (89)$$

$$= -\int_0^1 \int_1^2 e^{i(2-u)\overline{M}} \hat{F} e^{i(u-v)(\overline{M}+\hat{E})} \hat{F}^* e^{iv\overline{M}}\, dudv \quad (90)$$

The change of variable $uv \mapsto v$ gives

$$e^{i\overline{M}} \int_0^1 \int_0^1 u e^{i(1-u)\overline{M}} \hat{F} e^{iu(1-v)\overline{M}} \hat{F}^* e^{iuv\overline{M}}\, dudv$$

$$= \int_0^1 \int_0^u e^{i(2-u)\overline{M}} \hat{F} e^{i(u-v)(\overline{M}+\hat{E})} \hat{F}^* e^{iv\overline{M}}\, dvdu \quad (91)$$

Similarly

$$\left(\int_0^1 \int_0^1 u e^{i(1-u)\overline{M}} \hat{F} e^{iu(1-v)\overline{M}} \hat{F}^* e^{iuv\overline{M}}\, dudv\right) e^{i\overline{M}}$$

$$= \int_0^1 \int_0^u e^{i(1-u)\overline{M}} \hat{F} e^{i(u-v)(\overline{M}+\hat{E})} \hat{F}^* e^{i(1+v)\overline{M}}\, dvdu$$

$$= \int_1^2 \int_1^u e^{i(2-u)\overline{M}} \hat{F} e^{i(u-v)(\overline{M}+\hat{E})} \hat{F}^* e^{iv\overline{M}}\, dvdu \quad (92)$$

Subtracting Equations (91) and (92) from Equation (90), the contribution of $\hat{F}$ and $\hat{F}^*$ to the coefficient of dt is

$$-\int_0^2 \int_0^u e^{i(2-u)\overline{M}} \hat{F} e^{i(u-v)(\overline{M}+\hat{E})} \hat{F}^* e^{iv\overline{M}}\, dvdu$$

$$= -\int_0^1 \int_0^u e^{i(1-u)(2\overline{M})} (2\hat{F}) e^{i(u-v)(2\overline{M}+2\hat{E})} (2\hat{F})^* e^{iv(2\overline{M})}\, dvdu \quad (93)$$

Adding this to the contribution of $\tilde{H}$ the coefficient of dt is $\hat{H}_{\exp(i2\overline{M})}$. It follows from Equation (88) that

$$e^{i2\overline{M}_t} = I + \int_0^t (\hat{E}_{\exp(i2\overline{M})}\, d\Lambda + \hat{F}_{\exp(i2\overline{M})}\, dA + \hat{G}_{\exp(i2\overline{M})}\, dA^\dagger + \hat{H}_{\exp(i2\overline{M})}\, ds). \quad (94)$$

and the quantum Duhamel formula is valid for $2M$. It follows by induction that the formula is valid for $2^k M$ for all $k \in \mathbb{N}$.

We have shown above that pM satisfies (86) if Therefore $2^k pM$ satisfies (86) for all $k \in \mathbb{N}$ and $-1 \le p \le 1$. Therefore (86) is valid for all real p. ∎

It follows from the concluding remarks of Section 5 that there is a large class of $(2k+1)$-diagonal quantum semimartingales whose integrands are constructed from L^2-kernels. These semimartingales satisfy the quantum Duhamel formula.

Corollary 10.4 *Let M be a symmetric quantum semimartingale*

$$\hat{M}_t = \int_0^t \hat{E}\, d\Lambda + \hat{F}\, dA + \hat{F}^*\, dA^\dagger + \hat{H}\, ds.$$

with corresponding cmx semimartingale

$$M_t = \int_0^t (E\, d\Lambda + F\, dA + F^*\, dA^\dagger + H\, ds).$$

If the integrands E, F, F^* and H are processes of $(2k+1)$-diagonal matrices then $\hat{M}$ is essentially self-adjoint with closure $\overline{M}$ and $p\overline{M}$ satisfies the quantum Duhamel formula for all real p.

Proof. If $\psi \in \mathfrak{H}^j$ then $E_t{}^i_j\psi = (\hat{E}_t\psi)^i$ and $\|E_t{}^i_j\psi\| = \left\|(\hat{E}_t\psi)^i\right\| \leq \left\|\hat{E}_t\psi\right\|$. Therefore $\|E_t{}^i_j\| \leq \|E_t\|$ for all $t \in [0,1]$ and $\mathcal{V}^i_j(E) \leq \|E\|_\infty$. Similarly $\mathcal{V}^i_j(F) \leq \|F\|_2$ and $\mathcal{V}^i_j(H) \leq \|H\|_1$ for all i,j. If $\xi = \|E\|_\infty + 2\,\|F\|_2 + \|H\|_1$ then $\mathcal{K}$ is $(2k+1)$-diagonal with $\mathcal{K}^i_j \leq \xi(i+j)$ for all i,j. It follows from Proposition 7.3 that $\ell_{00} \subset \mathcal{A}_\epsilon(\Xi)$ for $\epsilon = \xi^{-1}(4k^2+2k)^{-1}$.

The required result now follows from Proposition 7.3 and Corollary 10.3. ∎

The argument in Corollary 10.3 only uses the fact that (86) is valid whenever $-\epsilon \leq p \leq \epsilon$ and does not rely on the properties of $\mathcal{K}$. Therefore it may be used to prove the following theorem.

Theorem 10.5 *Let $\hat{M} = \{\hat{M}_t : t \in [0,1]\}$ be a symmetric quantum semimartingale. If $\hat{M}_t$ is essentially self-adjoint and the quantum Duhamel formula (86) is valid for $-1 \leq p \leq 1$ then it is valid for all real p.*

The *Brownian quantum semimartingale* $B = A + A^\dagger$ is an essentially self-adjoint process in $\mathcal{S}'$ which does not belong to $\mathcal{S}_{\rm sa}$. The operators B_t are unbounded with common core $\mathcal{D}(N^{1/2})$, where N is the number operator. Let $\mathcal{W} : L^2(\Omega, \mathbb{P}^W) \hookrightarrow\!\!\!\!\!\rightarrow \mathfrak{H}$ be the Wiener–Ito isomorphism. Let $\hat{B} = (\hat{B}_t : t \in [0,1])$ be Brownian motion. Abusing notation, let $\hat{B}_t$ also denote the unbounded self-adjoint operator in $L^2(\Omega, \mathbb{P}^W)$ of multiplication by $\hat{B}_t$. Then $\mathcal{W} \circ \hat{B}_t \circ \mathcal{W}^{-1} = \overline{B}_t$ for $t \in [0,1]$.

The quantum semimartingale $\hat{B} = A + A^\dagger$ with quadruple $(0, I, I, 0)$ satisfies the condition of Corollary 10.4 so that $\overline{B}$ satisfies the quantum Duhamel formula

$$e^{i\overline{B}_t} = I + i\int_0^t e^{i\overline{B}_s}\, dB_s - \frac{1}{2}\int_0^t e^{i\overline{B}_s}\, ds. \tag{95}$$

This is equivalent, *via* the Wiener–Ito isomorphism, to the classical Ito formula for the function $x \mapsto e^{ix}$ and classical Brownian motion $W = \{W_t : t \in [0,1]\}$:

$$e^{iW_t} = I + i\int_0^t e^{iW_s}\, dW_s - \frac{1}{2}\int_0^t e^{iW_s}\, ds,$$

11 Quantum Ito Formula

As in [28] the quantum Duhamel formula in Section 10 leads, *via* the Fourier functional calculus, to a functional Ito formula.

If $\hat{T}$ is a self-adjoint linear transformation in $\mathfrak{H}$ and f and its Fourier transform $\hat{f}$ lie in $L^1(\mathbb{R})$ then define $f(\hat{T})$ in $\mathcal{B}(\mathfrak{H})$ by the Bochner integral

$$f(\hat{T}) = \int_{-\infty}^{\infty} \hat{f}(p) e^{ip\hat{T}}\, dp. \tag{96}$$

This definition of $f(\hat{T})$ agrees with the usual definition *via* the functional calculus.

It follows from the proof of Corollary 10.4 and the remarks preceding it that there is a class of non-trivial quantum semimartingales, whose integrands are $(2k+1)$-diagonal processes defined by L^2-kernels, satisfying the conditions of the following theorem.

Recall that $C^{2+}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : p \mapsto p^2\hat{f}(p) \text{ belongs to } L^1(\mathbb{R})\}$.

Theorem 11.1 *Let $\hat{M} = \{\hat{M}_t : t \in [0,1]\}$ be a symmetric quantum semimartingale*

$$\hat{M}_t = \int_0^t (\hat{E}_s\, d\Lambda_s + \hat{F}_s\, dA_s + \hat{F}^*\, dA_s^\dagger + \hat{H}_s\, ds)$$

with corresponding symmetric cmx semimartingale $M = \{M_t : t \in [0,1]\}$*:*

$$M_t = \int_0^t (E_s \, d\Lambda_s + F_s \, dA_s + F^* \, dA_s^\dagger + H_s \, ds).$$

If there exists $\epsilon > 0$ *such that* $\ell_{00} \subset \mathcal{A}_\epsilon(\mathcal{K})$ *where* $\mathcal{K}$ *is the control matrix of* M *then* $\hat{M}$ *is essentially self-adjoint with core* $\mathfrak{H}_{00}$*.*

For each $f \in C^{2+}(\mathbb{R})$ *the process* $f(\overline{M}) = \{f(\overline{M}_t) : t \in [0,1]\}$ *defined by the formula* (96) *is a regular quantum semimartingale which satisfies the quantum Ito formula:*

$$f(\overline{M}_t) = f(0) + \int_0^t (\hat{E}_{f(M)} \, d\Lambda + \hat{F}_{f(M)} \, dA + \hat{G}_{f(M)} \, dA^\dagger + \hat{H}_{f(M)} \, ds) \qquad (97)$$

where

$$\begin{aligned}
\hat{E}_{f(M)} = \int_{-\infty}^{\infty} \hat{f}(p) \hat{E}_{\exp(ipM)} \, dp &= f(M + \hat{E}) - f(M) \\
\hat{F}_{f(M)} = \int_{-\infty}^{\infty} \hat{f}(p) \hat{F}_{\exp(ipM)} \, dp &= \int_{-\infty}^{\infty} \int_0^1 ip \hat{f}(p) e^{ip(1-u)M} \hat{F} e^{ipu(M+\hat{E})} \, du \, dp \\
\hat{G}_{f(M)} = \int_{-\infty}^{\infty} \hat{f}(p) \hat{G}_{\exp(ipM)} \, dp &= \int_{-\infty}^{\infty} \int_0^1 ip \hat{f}(p) e^{ip(1-u)(M+\hat{E})} \hat{F}^* e^{ipuM} \, du \, dp \\
\hat{H}^*_{f(M)} = \int_{-\infty}^{\infty} \hat{f}(p) \hat{H}_{\exp(ipM)} \, dp &= \int_{-\infty}^{\infty} \int_0^1 ip \hat{f}(p) e^{ip(1-u)M} \hat{H} e^{ipuM} \, du \, dp \\
&\quad - \int_{-\infty}^{\infty} \int_0^1 \int_0^1 p^2 \hat{f}(p) u e^{ip(1-u)M} \hat{F} e^{ipu(1-v)(M+\hat{E})} \hat{F}^* e^{ipuvM} \, du \, dv \, dp
\end{aligned}$$

Proof. It follows from Corollary 10.3 that

$$\begin{aligned}
f(M_t) &= f(0) + \int_{-\infty}^{\infty} \hat{f}(p) \int_0^t (E_{\exp(ipM)} \, d\Lambda + F_{\exp(ipM)} \, dA \\
&\qquad + G_{\exp(ipM)} \, dA^\dagger + H_{\exp(ipM)} \, ds) \, dp. \qquad (98)
\end{aligned}$$

so that, formally, the theorem may be proved by interchanging the order of integration in (98). This requires a Fubini type theorem for regular quantum semimartingales. The proof is the same as when M itself is a regular quantum semimartingale [28, Theorem 6.1, Theorem 6.2]. ∎

Remark. The proof of [28, Theorem 6.1] is correct for the functions e, f, g, h defined by the formulae [28, (48)]. The proof under the given conditions is not quite correct. The sixth equality in that proof is not necessarily valid under hypothesis (d). However this may be remedied by observing that in the seventh equality the integral with respect to p may also be taken outside of the inner product.

In order to mimic the classical Ito formula we recast Equation 98 in terms of the differential $Df(M)(\cdot)$ and the Ito second differential $D_I^2 f(M)(\cdot,\cdot)$ of f at M.

If $\hat{H} \in \mathcal{B}(\mathfrak{H})$ is a bounded self-adjoint transformation then standard perturbation theory (Hille-Phillips)[10] shows that $i(\hat{T}+\hat{H})$ is the generator of a one parameter unitary group $p \mapsto e^{ip(\hat{T}+\hat{H})}$ so that $f(\hat{T}+\hat{H})$ is defined by (96) with $\hat{T}$ replaced by $\hat{T}+\hat{H}$. We show that f has a differential at $\hat{T}$ in the direction of H.

Proposition 11.2 *Let $\hat{T}$ be a self-adjoint operator and let $\hat{H}$ and $\hat{K}$ be bounded self-adjoint operator in $\mathfrak{H}$. If $f \in C^{2+}(\mathbb{R})$ then the first differential $Df(\hat{T})(\hat{H})$ and the second differential $D^2 f(\hat{T})(\hat{H},\hat{K})$ both exist and are given by the formulae*

$$Df(\hat{T})(\hat{H}) = \int_{-\infty}^{\infty}\int_0^1 ip\hat{f}(p)e^{ip(1-u)\hat{T}}\hat{H}e^{ipu\hat{T}}\,dudp; \tag{99}$$

$$\begin{aligned} Df(\hat{T})(\hat{H},\hat{K}) = \int_{-\infty}^{\infty}\int_0^1\int_0^1 -p^2\hat{f}(p)\Big(&e^{ip(1-u)\hat{T}}\hat{H}e^{ipu(1-v)\hat{T}}\hat{K}e^{ipuv\hat{T}} \\ &+e^{ip(1-u)\hat{T}}\hat{K}e^{ipu(1-v)\hat{T}}\hat{H}e^{ipuv\hat{T}}\Big)\,dudvdp. \end{aligned} \tag{100}$$

Proof. Consider the Duhamel formula

$$e^{i(\hat{T}+\epsilon\hat{H})} = e^{i\hat{T}}+\epsilon\int_0^1 e^{i(1-u)\hat{T}}\hat{H}e^{iu\hat{T}}\,du+\epsilon^2\int_0^1\int_0^1 e^{i(1-u)\hat{T}}\hat{H}e^{iu(1-v)\hat{T}}\hat{H}e^{iuv(\hat{T}+\epsilon\hat{H})}\,dudv.$$

Since $\left\|e^{i(1-u)\hat{T}}\right\|$, $\left\|e^{iu(1-v)\hat{T}}\right\|$ and $\left\|e^{iuv\hat{T}}\right\|$ are all 1 it follows that

$$\left\|\frac{e^{i(\hat{T}+\epsilon\hat{H})}-e^{i\hat{T}}}{\epsilon}-\int_0^1 e^{i(1-u)\hat{T}}\hat{H}e^{iu\hat{T}}\,du\right\| \le \epsilon\left\|\hat{H}\right\|^2.$$

Therefore

$$\left\| \frac{f(\hat{T}+\epsilon\hat{H})-f(\hat{T})}{\epsilon} - \int_{-\infty}^{\infty}\int_0^1 ip\hat{f}(p)e^{ip(1-u)\hat{T}}\hat{H}e^{ipu\hat{T}}\,dudp \right\| \leq \epsilon \left\|\hat{H}\right\|^2 \int_{-\infty}^{\infty} |p\hat{f}(p)|\,dp$$

which tends to zero with ϵ. This proves (99). A similar argument applied to the right hand side of Equation (99) may be used to prove (100). ∎

These differentials may be extended to all $\hat{H}$ and $\hat{K}$ in $\mathcal{B}(\mathfrak{H})$ as follows: If $\hat{H} = -\hat{H}^*$ and $\hat{K} = -\hat{K}^*$ then let

$$\begin{aligned} iDf(\hat{T})(\hat{H}) &= Df(\hat{T})(i\hat{H}); \\ D^2f(\hat{T})(\hat{H},\hat{K}) = iD^2f(\hat{T})(\hat{H},i\hat{K}) &= D^2f(\hat{T})(i\hat{H},i\hat{K}) = iD^2f(i\hat{H},\hat{K}). \end{aligned}$$

Each operator in $\mathcal{B}(\mathfrak{H})$ is, uniquely, the sum of a self-adjoint and skew adjoint transformation so these definitions extend uniquely by linearity and bilinearity. The *Ito second differential* , D_I^2, is defined *via* the matrix operator equation

$$\begin{aligned} &\frac{1}{2}D^2f\left(\begin{bmatrix} \hat{X} & 0 \\ 0 & \hat{X} \end{bmatrix}\right)\left(\begin{bmatrix} 0 & \hat{H} \\ \hat{K} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \hat{H} \\ \hat{K} & 0 \end{bmatrix}\right) \\ &\qquad = \begin{bmatrix} D_I^2f(\hat{X})(\hat{H},\hat{K}) & 0 \\ 0 & D_I^2f(\hat{X})(\hat{K},\hat{H}) \end{bmatrix}. \end{aligned} \tag{101}$$

D_I^2f is an unsymmetrised version of D^2f and $D^2f(\hat{H},\hat{K}) = D_I^2f(\hat{H},\hat{K}) + D_I^2f(\hat{K},\hat{H})$ [28, §2.].

We use the following notation:

$$\begin{aligned} Df(\overline{M})(d\hat{M}) &= Df(\overline{M})(\hat{E})\,d\Lambda + Df(\overline{M})(\hat{F})\,dA \\ &\qquad + Df(\overline{M})(\hat{G})\,dA^\dagger + Df(\overline{M})(\hat{H})\,dt, \\ D_I^2f(\overline{M})(d\hat{M},d\hat{M}) &= D_I^2f(\overline{M})(\hat{E},\hat{E})\,d\Lambda + D_I^2f(\hat{M})(\hat{F},\hat{E})\,dA \\ &\qquad + D_I^2f(\overline{M})(\hat{E},\hat{G})\,dA^\dagger + D_I^2f(\overline{M})(\hat{F},\hat{G})\,dt \end{aligned}$$

The quantum Ito formula is most heuristically satisfying when $\hat{E} \equiv 0$. In this case it is a generalisation of the classical Ito formula for Brownian (continuous) semimartingales. If $\hat{E} \not\equiv 0$ the more complicated formula [28, Theorem 6.2.], which generalises the classical Ito formula for some discontinuous semimartingales, is also true.

Corollary 11.3 (Quantum Ito Formula) *Suppose the symmetric quantum semimartingale*

$$\hat{M}_t = \int_0^t (\hat{F}_s \, dA_s + \hat{F}^* \, dA_s^\dagger + \hat{H}_s \, ds)$$

satisfies the conditions of Theorem 11.1.

If $f \in C^{2+}(\mathbb{R})$ *then* $f(\overline{M}) = \{f(\overline{M}_t) : t \in [0,1]\}$ *is a regular quantum semimartingale and satisfies the quantum Ito formula*

$$f(\overline{M}_t) \;\; = \;\; f(0) + \int_0^t (Df(\overline{M}_s)(d\hat{M}_s) + D_I^2 f(\overline{M}_s)(d\hat{M}_s, d\hat{M}_s)) \quad (102)$$

Proof. This theorem is a translation of Theorem 11.1 using the above terminology together with Proposition 11.2 ∎

The Brownian quantum semimartingale $\hat{B}_t = A_t + A_t^\dagger$, satisfies the conditions of Corollary 11.3. It follows from (102), or directly form Equation (95), that

$$f(\overline{B}_t) = f(0) + \int_0^t f'(\overline{B}_s) \, d\hat{B}_s + \frac{1}{2} \int_0^t f''(\overline{B}_s) \, ds$$

whenever $f \in C^{2+}(\mathbb{R})$. The analysis in [28, Section 7] and [1] shows that Equation (102) is equivalent to the classical Ito formula for Brownian motion

$$f(W_t) = f(0) + \int_0^t f'(W_s) \, dW_s + \frac{1}{2} \int_0^t f''(W_s) \, ds$$

Thus we have extended the quantum Ito formula for regular quantum semimartingales to a non-trivial class of essentially self adjoint quantum semimartingales containing the Brownian quantum semimartingale.

Example 11.4

For each real θ the quantum semimartingale $P_t(\theta) = e^{i\theta(t)} A_t + e^{-i\theta(t)} A_t^\dagger$ is representable as a Brownian motion. It has a *conjugate* quantum semimartingale $Q_t(\theta) = i(e^{i\theta(t)} A_t - e^{-i\theta(t)} A_t^\dagger)$, sometimes called the "upside-down" quantum Brownian motion. $Q(\theta)$ is the conjugate of $P(\theta)$ by the Fourier–Wiener transform: $Q_t(\theta) = \mathcal{F} P_t(\theta) \mathcal{F}^{-1}$ [16, IV §2.3, VI §1.13].

The case where $\theta(t)$ is a differentiable function of t is interesting. Let $P_t = P_t(\theta(t))$ and $Q_t = Q_t(\theta(t))$. Then $P = \{P_t : t \in [0,1]\}$ and $Q = \{Q_t : t \in [0,1]\}$ are essentially self-adjoint quantum semimartingales with core $\mathcal{E}$. Using the characterisation (69) of quantum stochastic integrals it is easy to check that

$$dP_t = e^{i\theta(t)}dA_t + e^{-i\theta(t)}dA_t^\dagger + \left(e^{i\theta(t)}A_t - e^{-i\theta(t)}A_t^\dagger\right)\theta'(t)\,dt.$$

P satisfies the conditions of Corollary 10.4. The differential form of the quantum Duhamel formula is

$$\begin{aligned} de^{iP_t} &= e^{iP_t}dP_t - \frac{1}{2}e^{iP_t}\,dt + i\theta'(t)\int_0^1 e^{i(1-u)P_t}(e^{i\theta(t)}A_t - e^{-i\theta(t)}A_t^\dagger)e^{iuP_t}du\,dt \\ &= e^{iP_t}dP_t - \frac{1}{2}e^{iP_t}\,dt + \theta'(t)\int_0^1 e^{i(1-u)P_t}Q_te^{iuP_t}du\,dt. \end{aligned}$$

The quantum Ito formula for $f \in C^{2+}(\mathbb{R})$ is

$$d(f(\hat{P}_t)) = f'(\hat{P}_s)\,d\hat{P}_s + \frac{1}{2}f''(\hat{P}_s)\,ds + \theta'(s)Df(P_s)(Q_s)ds$$

This is the classical Ito differential continuously perturbed by the last term which is $\theta'(t)$ times the differential of f at P_t with magnitude Q_t in the direction of the conjugate Brownian quantum semimartingale to P_t.

12 Quantum Stratonovich Formula.

Quantum Stratonovich product formulae arise naturally in the construction of models in quantum optics [8, Section II]. They also appear in the analysis of the quantum Liouville equation [26, *op. cit.*].

Care should be taken in interpreting the statement following [**?**, Equation 3.14.]. This could be thought to imply that the correction term in the functional is the symmetrised second differential rather than the correct unsymmetrised Ito second differential. This interpretation would lead to an incorrect statement of the functional quantum Stratonovich formula.

The functional quantum Stratonovich formula for regular quantum semimartingales may be deduced from the functional quantum Ito formula. The details will be published elsewhere.

Theorem 12.1 *Let*

$$\hat{M}_t = \int_0^t \hat{F}_s dA_s + \hat{F}_s^* \, dA_s^\dagger + \hat{H}_s \, ds$$

be a gauge-free regular self-adjoint quantum semimartingale. If $f \in C^{3+}(\mathbb{R})$ then

$$f(\hat{M}_t) = f(0) + \oint_0^t Df(\hat{M}_s)(d\hat{M}_s) \tag{103}$$

We have used the following notation.

$$C^{3+}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : p \mapsto p^3 \hat{f}(p) \text{ belongs to } L^1(\mathbb{R})\}.$$

The *quantum Stratonovich integral*

$$\oint_0^t Df(\hat{M}_s)(d\hat{M}_s) = \text{so}-\lim_{|\mathcal{P}|\to 0} \sum_{k=0}^{n-1} Df\left(\frac{1}{2}\left(\hat{M}_{t_{k+1}} + \hat{M}_{t_k}\right)\right)(\hat{M}_{t_{k+1}} - \hat{M}_{t_k})$$

where $\mathcal{P}$ is the partition $0 = t_0 < t_1 \cdots < t_n = t$ with $|\mathcal{P}| = \max\{|t_{k+1} - t_k| : k = 0, n-1\}$ and the convergence is in the strong operator topology.

Care should be taken in interpretation of the Stratonovich integral since, in general, $(A_{t_{k+1}} - A_{t_k})$ and $(A^\dagger_{t_{k+1}} - A^\dagger_{t_k})$ do not commute with $(\hat{M}_{t_{k+1}} + \hat{M}_{t_k})$ and

$$Df(\hat{M}_s)(d\hat{M}_s) \not\equiv Df(\hat{M}_s)(\hat{F}_s)\, dA_s + Df(\hat{M}_s)(\hat{F}_s^\dagger)\, dA_s^\dagger + Df(\hat{M}_s)(\hat{H}_s)\, ds.$$

The proof of Theorem 12.1 which uses the Taylor series expansion of $f : \mathcal{B}(\mathfrak{H}) \to \mathcal{B}(\mathfrak{H})$ to three terms with remainder depends on the regularity of $\hat{M}$. The control of the remainder when $\hat{M}$ is irregular becomes a serious obstruction.

The product formula may be recovered from the functional formula, with $f(x) = x^2$, using 2×2 matrix valued processes as in [28, §3].

It may also be shown that if $M = \int_0^t F_s dA_s + F_s^* \, dA_s^\dagger + H_s \, ds$ is a symmetric cmx semimartingale satisfying the conditions of Theorem 11.1 and f is a polynomial then

$$f(M_t) = f(0) + \oint_0^t Df(M_s)(dM_s) \tag{104}$$

where

$$\oint_0^t Df(M_s)(dM_s) = \text{se}-\lim_{|\mathcal{P}|\to 0} \sum_{k=0}^{n-1} Df\left(\frac{1}{2}\left(M_{t_{k+1}} + M_{t_k}\right)\right)\left(M_{t_{k+1}} - M_{t_k}\right)$$

and the se-limit is the *strong entrywise* limit: the limit in the strong operator topology in each matrix entry.

Problem 1. Is the cmx Stratonovich formula (104) valid for $f \in C^4(\mathbb{R}) \cap L^1(\mathbb{R})$ and does it give rise to a corresponding quantum Stratonovich formula ?

13 Perturbation of Classical Semimartingales

For the remainder of this article we will not be concerned with chaos matrices and the "ˆ" used to distinguish operator from chaos matrix will be dropped.

It follows from Section 5 and Corollary 10.4 that there is a large class of irregular essentially self-adjoint quantum semimartingales which satisfy the quantum Duhamel formula and consequently the quantum Ito formula. The next three sections are devoted to showing that this class is closed under perturbation by regular self-adjoint quantum semimartingales.

Theorem 13.1, the main result of this section, is a special case of Theorem 15.3. However the proof of the former is more sophisticated than the proof of the latter, which uses rather crude expansion methods.

Let $W = \{W_t : t \in [0,1]\}$ be Brownian motion on $(\Omega, \mathbb{P})$, the classical Wiener space for the time interval $[0,1]$ endowed with the Brownian filtration. If $f \in L^2[0,1]$ let $\psi(f)$ denote the exponential martingale:

$$\psi(f) = e^{\int_0^1 f(s)\, dW_s - \frac{1}{2}\int_0^1 f(s)^2\, ds}$$

If $\mathcal{W} : L^2(\Omega, \mathbb{P}) \hookrightarrow\!\!\!\!\rightarrow \mathfrak{H}$ is the Wiener–Ito isomorphism then $e(f) = \mathcal{W}\psi(f)$.

Let F be a bounded real-valued adapted stochastic process on $(\Omega, \mathbb{P})$. Then the martingale $\mathsf{M}_t = \int_0^t \mathsf{F}_s\, dW_s$ satisfies the classical Ito formula:

$$f(\mathsf{M}_t) \quad = \quad f(0) + \int_0^t f'(\mathsf{M}_s)\, d\mathsf{M}_s + \frac{1}{2}\int_0^t f''(\mathsf{M}_s)\, d\left\langle \mathsf{M}\right\rangle_s, \qquad (105)$$

where $d\mathsf{M}_s = \mathsf{F}_s\, dW_s$ and $d\left\langle \mathsf{M}\right\rangle_s = \mathsf{F}_s^2\, ds$.

If $\mathsf{F} = I$ then $M = \mathcal{W} \circ \mathsf{M} \circ \mathcal{W}^{-1}$ is the Brownian quantum semimartingale For more general F the cmx representation of $F = \mathcal{W} \circ \mathsf{F} \circ \mathcal{W}^{-1}$ is complicated and does not satisfy the conditions of Corollary 11.3 and the classical Ito formula (105) does not directly follow from (102).

Using a martingale moment inequality [12, Exercise 3.25]

$$\mathbb{E}[\mathsf{M}_t^4] \leq 6\mathbb{E}\left[\int_0^t |\mathsf{F}_s^4|\, ds\right] \leq 6\mathbb{E}\left[\int_0^1 |\mathsf{F}_s^4|\, ds\right]$$

so that $\mathsf{M}_t \in L^4(\Omega, \mathbb{P})$ for $t \in [0,1]$. It follows from direct calculation that $\psi(f) \in L^4(\Omega, \mathbb{P})$.

However M is a martingale and $M = \{M_t : t \in [0,1]\}$ defined by

$$M_t = \int_0^t F\, dA + F\, dA^\dagger$$

is a quantum semimartingale that satisfies the Ito formula

$$f(\overline{M}_t) = f(0) + \int_0^t f'(\overline{M}_s) F_s (dA_s + dA_s^\dagger) + \frac{1}{2} \int_0^t f''(M_s) F_s^2\, ds$$

for all bounded C^2 functions f. By [23, VIII.3 Proposition 2] with $p = 4$ it follows that for each $t \in [0,1]$ the operator of multiplication by M_t is self-adjoint on its maximal domain and that $\mathcal{E}_W$ is a core for M_t.

These facts may be used to show that perturbations of M by regular quantum semimartingales satisfy the quantum Ito formula.

Theorem 13.1 *Let $N = N(R, S + F, S^* + F, U)$ be the quantum semimartingale $M + J$ where J is a regular self-adjoint quantum semimartingale:*

$$J_t = \int_0^t (R\, d\Lambda + S\, dA + S^*\, dA^\dagger + U\, ds).$$

Then pN is essentially self-adjoint and satisfies the quantum Duhamel formula, (86)*, for each real p.*

Corollary 13.2 *If $f \in C^{2+}(\mathbb{R})$ then $f(\overline{N})$ satisfies the quantum Ito formula* (97)*.*

We will only outline the proof, which follows that for the special case $F = I$ in [29]. We shall need a result from that paper.

Proposition 13.3 *Let $N^{[n]} \equiv N^{[n]}(E^{[n]}, F^{[n]}, G^{[n]}, H^{[n]})$, be a sequence of regular quantum semimartingales and let $N = N(E, F, G, H)$ be a quantum semimartingale such that:*

(i) $E_t^{[n]}$, $F_t^{[n]}$, $G_t^{[n]}$ *and* $H_t^{[n]}$ *converge strongly to* E_t, F_t, G_t *and* H_t *respectively for almost all* $t \in [0, 1]$;

(ii) $\sup\{\|N_t^{[n]}\| : t \in [0, 1],\ n = 1, 2, \ldots\} = K < \infty$;

(iii) $\sup_n \|E_t^{[n]}\| = \alpha(t) \in L^\infty[0, 1]$;

(iv) $\sup_n(\|F_t^{[n]}\| + \|G_t^{[n]}\|) = \beta(t) \in L^2[0, 1]$;

(v) $\sup_n \|E_t^{[n]}\| = \gamma(t) \in L^1[0, 1]$.

Then

(a) N *is regular with* $\|N_t\| \le K$ *for all* $t \in [0, 1]$;

(b) $N_t^{[n]}$ *converges strongly to* N_t *for all* $t \in [0, 1]$.

We also use the following corollary of Trotter's theorem. It is a direct consequence of [23, Theorems VIII.21, VIII.25(a)].

Theorem 13.4 *Let T_n and T be self-adjoint operators and suppose that $\mathcal{D}$ is a common core for T and all T_n. If $T_n\varphi \to T\varphi$ for each φ in $\mathcal{D}$ then e^{ipT_n} converges strongly to e^{ipT} for each real p.*

Proof of Theorem 13.1. It is enough to consider the case $p = 1$ and $\left\|\hat{F}_t\right\| \le 1$ for all t. Then $N_t = M_t + J_t$ is essentially self–adjoint on $\mathcal{E}$ by the Kato–Rellich theorem [23, Theorem X.12]. Since J_t is bounded $\overline{N}_t = \overline{M}_t + \overline{J}_t$, where operators are added on their common domain.

For each $n = 1, 2, \ldots$ choose a C^∞ function $h_n : \mathbb{R} \to [0, \infty)$ with $h_n = 0$ on $(-\infty, -4n] \cup [4n, \infty)$, with $h_n(x) = x$ for $x \in [-n, n]$ and with $-1 \leq h_n', h_n'' \leq 1$. By the classical Ito formula [24, IV 3.]

$$h_n(\mathsf{M})_t = \int_0^t h_n'(\mathsf{M}_s) F_s \, dW_s + \frac{1}{2} \int_0^t h_n''(\mathsf{M}_s) \hat{F}_s^2 \, ds.$$

Moreover the multiplication operators $h_n(\mathsf{M})_t$, $h_n'(\mathsf{M})_t$ and $h_n''(\mathsf{M})_t$ are bounded with $\|h_n(\mathsf{M})_t\| \leq n$ and $\|h_n'(\mathsf{M})_t\|$, $\|h_n''(\mathsf{M})_t\| \leq 1$. Applying the Wiener–Ito isomorphism, the restriction of $h_n(\overline{M})$ to $\mathcal{E}$ is a regular essentially self-adjoint quantum semimartingale and, for $\psi \in \mathcal{E}$,

$$h_n(\overline{M})_t \psi = \int_0^t (h_n'(\overline{M}_s) F_s (dA_s + dA_s^\dagger) + \frac{1}{2} h_n''(\overline{M}_s) F_s^2 \, ds) \psi.$$

The processes

$$F^{(n)} = S + h_n'(\overline{M}) F, \quad G^{(n)} = S^* + h_n'(\overline{M}) F, \quad H^{(n)} = U + \frac{1}{2} h_n''(\overline{M}_s) F^2$$

are strongly measurable and

$$\left\| F_t^{(n)} \right\|, \left\| G_t^{(n)} \right\| \leq \|F_t\| + 1; \qquad \left\| H_t^{(n)} \right\| \leq \|U_t\| + 1/2;$$

$$(F^{(n)})^* = G^{(n)}; \qquad (H^{(n)})^* = H^{(n)}.$$

Therefore

$$N^{(n)} = \int_0^t (E_s \, d\Lambda_s + F_s^{(n)} \, dA_s + G_s^{(n)} \, dA_s^\dagger + H_s^{(n)} \, ds)$$

is a regular self-adjoint quantum semimartingale with

$$\overline{N}^{(n)} = \overline{J} + h_n(\overline{M})$$

Let $\psi \in \mathcal{E}$. Since $N^{(n)}$ and R are bounded $\mathcal{E}$ is a common core for M_t, N_t, $N_t^{(n)}$ and R_t for all $t \in [0, 1]$ and $n \in \mathbb{N}$. By the spectral theorem [23, Theorem VIII.5 (c)] $\lim_{n \to \infty} h_n(\overline{M}_t) \psi = \overline{M}_t \psi$. Since J is regular $N_t^{(n)} \varphi = h_n(\overline{M}_t) \varphi + \overline{J}_t \varphi$ converges to $\overline{N}_t \varphi$. Similarly $(N_t^{(n)} + R_t) \varphi$ converges to $(\overline{N}_t + R_t) \varphi$

It follows from Theorem 13.4 that, for each real p,

$$\lim_{n\to\infty} e^{ipN_t^{(n)}} = e^{ip\overline{N}_t}, \qquad \lim_{n\to\infty} e^{ip(N_t^{(n)}+R_t)} = e^{ip(\overline{N}_t+R_t)}, \tag{106}$$

in the strong operator topology.

Now $\|h_n'\|_\infty, \|h_n''\|_\infty \le 1$ and h_n' and h_n'' converge pointwise to 1 and 0 respectively. By the spectral theorem [23, Theorem VIII.5 (d)] $h_n'(\overline{M}_t)$ is strongly convergent to I and $h_n''(\overline{M}_t)$ is strongly convergent to 0. Therefore

$$\lim_{n\to\infty} F_t^{(n)} = S_t + F_t, \quad \lim_{n\to\infty} G_t^{(n)} = S_t^* + F_t, \quad \lim_{n\to\infty} H_t^{(n)} = U_t, \tag{107}$$

in the strong operator topology.

The regular quantum semimartingale $N^{(n)}$ satisfies the quantum Duhamel formula, [28, Proposition 5.1], for $p = 1$:

$$e^{iN_t^{(n)}} = I + \int_0^t (E^{[n]}\, d\Lambda + F^{[n]}\, dA + G^{[n]}\, dA^\dagger + H^{[n]}\, ds),$$

where

$$\begin{aligned}
E^{[n]} = E_{\exp(iN^{(n)})}(t) &= e^{i(N_t^{(n)}+R_t)} - e^{iN_t^{(n)}}, \\
F^{[n]} = F_{\exp(iN^{(n)})}(t) &= i\int_0^1 e^{i(1-u)N_t^{(n)}} F_t^{(n)} e^{iu(N_t^{(n)}+R_t)}\, du, \\
G^{[n]} = G_{\exp(iN^{(n)})}(t) &= i\int_0^1 e^{i(1-u)(N_t^{(n)}+R_t)} G_t^{(n)} e^{iuN_t^{(n)}}\, du, \\
H^{[n]} = H_{\exp(iN^{(n)})}(t) &= i\int_0^1 e^{i(1-u)N_t^{(n)}} H_t^{(n)} e^{iuN_t^{(n)}}\, du \\
&\quad - \int_0^1 \int_0^1 u e^{i(1-u)N_t^{(n)}} F_t^{(n)} e^{iu(1-v)(N_t^{(n)}+R_t)} G_t^{(n)} e^{iuvN_t^{(n)}}\, du\, dv.
\end{aligned}$$

If $t, u \in [0,1]$ and $n \in \mathbb{N}$ then

$$\left\| e^{i(1-u)N_t^{(n)}} F_t^{(n)} e^{iu(N_t^{(n)}+R_t)} \right\| \le \left\| F_t^{(n)} \right\| \le \|F_t\| + 1.$$

Let $\psi \in \mathfrak{H}$. Since operator multiplication is continuous on norm bounded sets for the strong operator topology [4, Proposition 2.4.1.]

$$\lim_{n\to\infty} e^{i(1-u)N_t^{(n)}} F_t^{(n)} e^{iu(N_t^{(n)}+R_t)}\psi = e^{i(1-u)\overline{N}_t} F_t^{(n)} e^{iu(\overline{N}_t+R_t)}\psi.$$

By the dominated convergence theorem for Bochner integrals [6, Chapter II, Theorem 3]

$$\lim_{n\to\infty} F_{\exp(iN^{(n)})}(t)\psi = \lim_{n\to\infty} F_{\exp(i\overline{N})}(t)\psi.$$

Similarly $E_{\exp(iN^{(n)})}(t)$, $G_{\exp(iN^{(n)})}(t)$ and $H_{\exp(iN^{(n)})}(t)$ converge strongly to $E_{\exp(i\overline{N})}(t)$, $G_{\exp(i\overline{N})}(t)$ and $H_{\exp(i\overline{N})}(t)$ respectively, where the limits are defined by (87). Therefore condition (i) of Proposition 13.3 is satisfied by the quantum semimartingales $e^{i\overline{N}}$ and $e^{iN^{(n)}}$.

Condition (ii) of Proposition 13.3 is satisfied since $e^{iN_t^{(n)}}$ is unitary.

Since $\|f_n'(M_t)\|$, $\|f_n''(M_t)\| \leq 1$ the $F^{(n)}, G^{(n)} \in L^2([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}})$ and $H^{(n)} \in L^1([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}})$. Since $J(R, S, S^*, U)$ is a quantum semimartingale

$$\begin{aligned}
\left\|E_{\exp(iN^{(n)})}(t)\right\| &\leq 2 = \alpha(t) \in L^\infty[0,1], \\
\left\|F_{\exp(iN^{(n)})}(t)\right\| + \left\|G_{\exp(iN^{(n)})}(t)\right\| &\leq \left\|F_t^{(n)}\right\| + \left\|G_t^{(n)}\right\| = \beta(t) \in L^2[0,1], \\
\left\|H_{\exp(iN^{(n)})}(t)\right\| &\leq \left\|H_t^{(n)}\right\| + \left\|F_t^{(n)}\right\| \left\|G_t^{(n)}\right\| = \gamma(t) \in L^1[0,1].
\end{aligned}$$

and conditions (iii), (iv) and (v) of Proposition 13.3 are satisfied.

It follows from Proposition 13.3 that $e^{i\overline{N}}$ is a regular quantum semimartingale and satisfies the quantum Duhamel formula (86). This outlines the proof of Theorem 13.1. ∎

Corollary 13.2 follows from Theorem 13.1 exactly as Theorem 11.1 follows from Corollary 10.3.

14 The Duhamel Expansion

Another way to show that a perturbed quantum semimartingales satisfies the quantum Duhamel formula is to expand both sides of the formula using the classical Duhamel expansion.

Let $J, J_1, \ldots, J_n$ be bounded self-adjoint operators in a Hilbert space H and let M be a self-adjoint operator in H. Define the kernel

$$K^{(n)}(M : J_1, \ldots, J_n) \stackrel{\text{def}}{=\!=} \int_{\Delta^{(n)}} e^{i(1-u_1)M} J_1 e^{i(u_1-u_2)M} J_2 \cdots e^{i(u_{n-1}-u_n)M} J_n e^{iu_nM} du$$

Where $\Delta^{(n)} = \{1 \geq u_1 \geq \cdots u_j \geq \cdots \geq u_n \geq 0\}$ and $du = du_1 \ldots du_n$.

Then $K^{(n)}(M : J_1, \ldots, J_n)$ is a bounded linear operator and

$$\left\| K^{(n)}(M : J_1, \ldots, J_n) \right\| \;\leq\; \frac{\|J_1\| \cdots \|J_n\|}{n!} \tag{108}$$

Now put $K^{(0)}(M, J) = e^{iM}$ and $K^{(n)}(M, J) = K^{(n)}(M : J, \ldots, J)$.

Using the bounds (108) it may be shown that

$$e^{i(M+J)} = \sum_{n=0}^{\infty} i^n K^{(n)}(M, J), \tag{109}$$

the series being uniformly convergent in $\mathcal{B}(H)$. The expansion (109) is known as the Duhamel expansion of $e^{i(M+J)}$. A proof may be found in [10].

If $\Phi_1, \Phi_2, \ldots$ belong to $\mathcal{B}(H)$ and $k \leq n$ then put

$$K^{(n)}(M, J : \overset{(j_1)}{\Phi_1}, \ldots, \overset{(j_k)}{\Phi_k}) = K^{(n)}(M : J_1, \ldots, J_n)$$

where

$$J_j = \begin{cases} \Phi_i & \text{if } j = j_i,\ i = 1, \ldots, k, \\ J & \text{otherwise.} \end{cases}$$

This definition is independent of the order of the Φs: if τ is a permutation of $\{1, \ldots, k\}$ then Thus

$$K^{(n)}(M, J : \overset{(j_{\tau(1)})}{\Phi}{}_{\tau(1)}, \ldots, \overset{(j_{\tau(k)})}{\Phi}{}_{\tau(k)}) = K^{(n)}(M, J : \overset{(j_1)}{\Phi_1}, \ldots, \overset{(j_k)}{\Phi_k})$$

We sometimes compress this notation by suppressing M and J. Thus

$$\begin{aligned} K^{(n)} &= K^{(n)}(M, J), \\ K^{(n)}(w) &= K^{(n)}(wM, wJ), \\ K^{(n)}(\overset{(j_1)}{\Phi_1}, \ldots, \overset{(j_k)}{\Phi_k}) &= K^{(n)}(M, J : \overset{(j_1)}{\Phi_1}, \ldots, \overset{(j_k)}{\Phi_k}) \\ K^{(n)}(w; \overset{(j_1)}{\Phi_1}, \ldots, \overset{(j_k)}{\Phi_k}) &= K^{(n)}(wM, wJ : w\overset{(j_1)}{\Phi_1}, \ldots, w\overset{(j_k)}{\Phi_k}) \end{aligned}$$

If $\alpha_1, \alpha_2, \alpha_3, \ldots$ are positive integers and $\sigma_j = \alpha_1 + \cdots + \alpha_j + j$ then

$$K^{(\sigma_2 - 1)}(\overset{(\sigma_1)}{\Phi_1}) \;=\; \int_0^1 (1-w)^{\alpha_1} w_1^{\alpha_2} K^{(\alpha_1)}(1-w)\Phi_1 K^{(\alpha_2)}(w)\, dw \tag{110}$$

$$K^{(\sigma_3-1)}(\overset{(\sigma_1)}{\Phi_1}, \overset{(\sigma_2)}{\Phi_2}) \tag{111}$$
$$= \int_{\Delta^{(2)}} (1-w_1)^{\alpha_1}(w_1-w_2)^{\alpha_2} w_2^{\gamma} K^{(\alpha_1)}(1-w_1)\Phi_1 K^{(\alpha_2)}(w_1-w_2)\Phi_2 K^{(\alpha_3)}(w_2)\, dw$$

These identities are special cases of the following proposition.

Proposition 14.1 *If $\alpha_1, \ldots, \alpha_n, \ldots$ are positive integers and $\sigma_j = \alpha_1 + \cdots + \alpha_j + j$ then*

$$K^{(\sigma_{n+1}-1)}(\overset{(\sigma_1)}{\Phi_1}, \ldots, \overset{(\sigma_j+j)}{\Phi}_j, \ldots, \overset{(\sigma_n)}{\Phi_n}) \tag{112}$$
$$= \int_{\Delta^{(n)}} (1-w_1)^{\alpha_1} \cdots (w_{j-1}-w_j)^{\alpha_j} \cdots (w_{n-1}-w_n)^{\alpha_n} w_n^{\alpha_{n+1}} K^{(\alpha_1)}(1-w_1)\Phi_1 \cdots$$
$$\cdots \Phi_{j-1} K^{(\alpha_j)}(w_{j-1}-w_j)\Phi_j \cdots \Phi_n K^{(\alpha_{n+1})}(w_n)\, dw$$

Proof. The right hand side of (112) is

$$\int_{\Delta^{(n)}\times\Delta^{(\alpha_1)}\times\cdots\Delta^{(\alpha_n)}} (1-w_1)^{\alpha_1}\cdots(w_{n-1}-w_n)^{\alpha_n} w_n^{\alpha_{n+1}} e^{i(1-u_1^1)(1-w_1)M} J \cdots$$
$$\cdots J e^{iu^1_{\alpha_1}(1-w_1)M}\Phi_1 e^{i(1-u_1^2)(w_1-w_2)M} J \cdots J e^{iu^2_{\alpha_2})(w_1-w_2)M}\Phi_2 e^{i(1-u_1^3)(w_2-w_3)M} J \cdots$$
$$\cdots J e^{iu^j_{\alpha_j})(w_{j-1}-w_j)M}\Phi_j e^{i(1-u_1^{j+1})(w_j-w_{j+1})M} J \cdots$$
$$\cdots J e^{iu^{j+1}_{\alpha_{j+1}}(w_j-w_{j+1})M}\Phi_{j+1} e^{i(1-u_1^{j+2})(w_{j+1}-w_{j+2})M} J \cdots$$
$$\cdots J e^{iu^n_{\alpha_n}(w_{n-1}-w_n)M}\Phi_n e^{i(1-u_1^{n+1})w_n M} J \cdots J e^{iu^{n+1}_{\alpha_{n+1}} w_{n+1} M}\, dw du^1 \ldots du^{n+1}$$

By applying the change of variables

$$\begin{array}{lcll} \xi_j^{(k)} & = & w_k + u_j^{(k)}(w_{k-1}-w_k) & w_0 = 1;\ k = 1,\ldots,n;\ j = 1,\ldots,\alpha_k, \\ \xi^{(k)}_{\alpha_k+1} & = & w_k & k = 1,\ldots,n, \\ \xi_j^{(n+1)} & = & u_j^{(n+1)} w_n & j = 1,\ldots,\alpha_{n+1}, \end{array}$$

we obtain the left hand side of (112). ∎

We will also need a second order Duhamel expansion. If J and R belong to $\mathcal{B}(H)$ then

$$e^{i(M+J+R)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{1\le j_1<\cdots<j_k\le n} K^{(n)}(\overset{(j_1)}{R}, \ldots, \overset{(j_1)}{R})$$

If $R = 0$ this reduces to (109) so that

$$e^{i(M+J+R)} - e^{i(M+J)} = \sum_{n=0}^{\infty}\sum_{k=1}^{n}\sum_{1\le j_1<\cdots<j_k\le n} K^{(n)}(\overset{(j_1)}{R}, \ldots, \overset{(j_1)}{R}) \tag{113}$$

15 Perturbation of Quantum Semimartingales

We shall need a series version of Proposition 13.3 and a lemma on quantum stochastic differentiation under the integral sign.

Proposition 15.1 *Let $K^{(n)} \equiv K^{(n)}(E^{(n)}, F^{(n)}, G^{(n)}, H^{(n)})$, be a sequence of regular quantum semimartingales and let $N = N(E, F, G, H)$ be a quantum semimartingale such that:*

(i) $\sum_{n=0}^{\infty} E_t^{(n)}$, $\sum_{n=0}^{\infty} F_t^{(n)}$, $\sum_{n=0}^{\infty} G_t^{(n)}$ *and* $\sum_{n=0}^{\infty} H_t^{(n)}$ *converge strongly to* E_t, F_t, G_t *and* H_t *respectively for almost all* $t \in [0,1]$;

(ii) $\sum_{n=0}^{\infty} \|K_t^{(n)}\| = \kappa(t) \in L^{\infty}[0,1]$;

(iii) $\sum_{n=0}^{\infty} \|E_t^{(n)}\| = \alpha(t) \in L^{\infty}[0,1]$;

(iv) $\sum_{n=0}^{\infty}(\|F_t^{(n)}\| + \|G_t^{(n)}\|) = \beta(t) \in L^2[0,1]$;

(v) $\sum_{n=0}^{\infty} \|E_t^{(n)}\| = \gamma(t) \in L^1[0,1]$.

Then

(a) N *is regular with* $\|N_t\| \leq \kappa$ *for all* $t \in [0,1]$;

(b) $\sum_{n=0}^{\infty} K_t^{(n)}$ *converges strongly to* N_t *for all* $t \in [0,1]$.

Lemma 15.2 *For each $u \in \Delta^{(n)}$ let*

$$k_t(u) = \int_0^t \textsc{e}(u)\, d\Lambda + \textsc{f}(u)\, dA + \textsc{g}(u)\, dA^{\dagger} + \textsc{h}(u)\, ds$$

be a regular quantum semimartingale with $\|K_t(u)\| \leq c < \infty$ for all $t \in [0,1]$ and $u \in \Delta$. Suppose also that $u \mapsto k(u)$, $\textsc{e}(u)$, $u \mapsto \textsc{f}(u)$, $\textsc{g}(u)$ and $u \mapsto \textsc{h}(u)$ are strong operator continuous maps from $\Delta^{(n)}$ to $L^{\infty}([0,1], \mathcal{B}(\mathfrak{H})_{\mathrm{so}})$, $L^2([0,1], \mathcal{B}(\mathfrak{H})_{\mathrm{so}})$ and $L^1([0,1], \mathcal{B}(\mathfrak{H})_{\mathrm{so}})$ respectively. Then $K = \{K_t : t \in [0,1\}$ defined by

$$K_t = \int_{\Delta} k_t(u)\, du,$$

is a regular quantum semimartingale. Moreover

$$K_t = \int_0^t (E\,d\Lambda + F\,dA + G\,dA^\dagger + H\,ds), \tag{114}$$

where

$$E = \int_{\Delta^{(n)}} \mathrm{E}(u)\,du, \quad F = \int_{\Delta^{(n)}} \mathrm{F}(u)\,du$$
$$G = \int_{\Delta^{(n)}} \mathrm{G}(u)\,du, \quad H = \int_{\Delta^{(n)}} \mathrm{H}(u)\,du$$

Proof. It follows from (69) that

$$\begin{aligned}\langle K_t(u)e(f), e(g)\rangle &= \int_{\Delta^{(n)}} \langle k_t(u)e(f), e(g)\rangle \, du \\ &= \int_{\Delta^{(n)}} \int_0^t \langle fg\mathrm{E}(u) + f\mathrm{F}(u) + \overline{g}\mathrm{G}(u) + \mathrm{H}(u))(s)e(f), e(g)\rangle \, ds \\ &= \int_0^t \langle fgE(u) + fF(u) + \overline{g}G(u) + H(u))(s)e(f), e(g)\rangle \, ds.\end{aligned}$$

The conditions of the theorem have been chosen so that the interchange of integrals is valid. The identity (114) now follows from the characterisation of quantum stochastic integrals (69). ∎

Although the following theorem is true as stated below we shall only prove it when $E = 0$.

Theorem 15.3 *Let $M = M(E, F, F^*, H)$ be and essentially self-adjoint quantum semimartingale which satisfies the quantum Duhamel formula and let $J = J(R, S, S^*, U)$ be a regular quantum semimartingale. Then $M + J$ is an essentially self-adjoint quantum semimartingale which satisfies the quantum Duhamel formula.*

Proof. By the Kato–Rellich theorem $M + J$ is an essentially self-adjoint quantum semimartingale.

To prove the quantum Duhamel formula we formally expand the terms on both sides using the Duhamel expansion and equate the resulting expressions. The formal calculations are then made rigorous.

The Duhamel expansion (109) of $e^{i(M_t+J_t)}$ is

$$e^{i(M_t+J_t)} = \sum_{n=0}^{\infty} i^n K^{(n)}(M_t, J_t) \tag{115}$$

where $K^{(n)} = K^{(n)}(M, J)$ and

$$K^{(n)} = \int_{\Delta^{(n)}} e^{i(1-u_1)M} J e^{i(u_1-u_2)M} \cdots e^{i(u_{n-1}-u_n)M} J e^{iu_n M}\, du.$$

The integrand $k^{(n)}$ is the product of elements of $\mathcal{S}$ and so belongs to $\mathcal{S}$. The quantum stochastic differential $dk^{(n)}$ may be calculated from the quantum Duhamel formula and the quantum Ito product formula (this is done in the computation of $dK^{(n)}$ below.) Since multiplication is continuous on bounded sets for the strong operator topology it follows that $k^{(n)}$ satisfies the conditions of Lemma 15.2. Thus $K^{(n)}$ is in $\mathcal{S}$ and its differential may be calculated using the formula (114).

The regular quantum semimartingale $t \mapsto \int_0^t H_s\, ds$ may be incorporated in J so that H may be assumed to be zero. To further reduce the notational complexity we proceed in two steps.

Step 1. Assume that the coefficients E, H and R are zero. It then follows from the quantum Ito product formula that the product of any three quantum differentials taken from $\{dM, dJ\}$ is zero. Since dJ and dM have no terms in $d\Lambda$ we must prove the simple form of the quantum Duhamel formula:

$$\begin{aligned} e^{i(M_t+J_t)} &= I + \int_0^t i \int_0^1 e^{i(1-u)(M+J)} d(M+J) e^{iu(M+J)}\, du \qquad (116) \\ &+ \int_0^t i^2 \int_{\Delta^{(2)}} e^{i(1-u_1)(M+J)} d(M+J) e^{i(u_1-u_2)(M+J)} d(M+J) e^{iu_2(M+J)}\, du \end{aligned}$$

Fixing $u_0 = 1$ and $u_{n+1} = 0$ we have, by formula (114),

$$\begin{aligned} dK^{(n)} &= \sum_{j=1}^{n} \int_{\Delta^{(n)}} e^{i(1-u_1)M} J \cdots e^{i(u_{j-1}-u_j)M} dJ \cdots J e^{iu_n M}\, du \\ &+ \sum_{1\le j<k\le n} \int_{\Delta^{(n)}} e^{i(1-u_1)M} J \cdots e^{i(u_{j-1}-u_j)M} dJ \cdots e^{i(u_{k-1}-u_k)M} dJ \cdots J e^{iu_n M}\, du \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n+1}\int_{\Delta^{(n)}} e^{i(1-u_1)M} J\cdots d\left(e^{i(u_{j-1}-u_j)M}\right) J\cdots J e^{iu_nM}\,du \\
&+\sum_{1\le j<k\le n}\int_{\Delta^{(n)}} e^{i(1-u_1)M} J\cdots e^{i(u_{j-1}-u_j)M} dJ\cdots d\left(e^{i(u_{k-1}-u_k)M}\right) J\cdots J e^{iu_nM}\,du \\
&+\sum_{1\le j<k\le n}\int_{\Delta^{(n)}} e^{i(1-u_1)M} J\cdots d\left(e^{i(u_{j-1}-u_j)M}\right) J\cdots e^{i(u_{k-1}-u_k)M} dJ\cdots J e^{iu_nM}\,du \\
&+\sum_{1\le j<k\le n+1}\int_{\Delta^{(n)}} e^{i(1-u_1)M} J\cdots d\left(e^{i(u_{j-1}-u_j)M}\right) J\cdots d\left(e^{i(u_{k-1}-u_k)M}\right) J\cdots J e^{iu_nM}\,du \\
&=\sum_{j=1}^{n} K^{(n)}(\overset{(j)}{dJ}) + \sum_{1\le j<k\le n} K^{(n)}(\overset{(j)}{dJ},\overset{(k)}{dJ}) + \sum_{j=1}^{n+1}\left(iK^{(n+1)}(\overset{(j)}{dM}) + i^2K^{(n+2)}(\overset{(j)}{dM},\overset{(j+1)}{dM})\right) \\
&+\sum_{1\le j<k\le n+1} iK^{(n+1)}(\overset{(j)}{dJ},\overset{(k)}{dM}) + \sum_{1\le j<k\le n+1} iK^{(n+1)}(\overset{(j)}{dM},\overset{(k)}{dJ}) \\
&+\sum_{1\le j<k\le n+1} i^2K^{(n+2)}(\overset{(j)}{dM},\overset{(k+1)}{dM}).
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{j=1}^{n} K^{(n)}(\overset{(j)}{dJ}) + \sum_{1\le j<k\le n} K^{(n)}(\overset{(j)}{dJ},\overset{(k)}{dJ}) && (117)\\
&+\sum_{j=1}^{n+1} iK^{(n+1)}(\overset{(j)}{F^*})dA^\dagger + \sum_{1\le j<k\le n+1} iK^{(n+1)}(\overset{(j)}{dJ},\overset{(k)}{dM}) && (118)\\
&+\sum_{j=1}^{n+1} iK^{(n+1)}(\overset{(j)}{F})dA + \sum_{1\le j<k\le n+1} iK^{(n+1)}(\overset{(j)}{dM},\overset{(k)}{dJ}) && (119)\\
&+\sum_{1\le j<k\le n+2} i^2K^{(n+2)}(\overset{(j)}{dM},\overset{(k)}{dM}). && (120)
\end{aligned}
$$

The terms above may be calculated by using the multilinearity of the $K^{(n)}$ then taking out the basic differentials preserving their order and then using the quantum Ito table. For example

$$
\begin{aligned}
K^{(n)}(\overset{(j)}{dJ}) &= K^{(n)}(\overset{(j)}{S})\,dA + K^{(n)}(\overset{(j)}{S^*})\,dA^\dagger + K^{(n)}(\overset{(j)}{R})\,dt,\\
K^{(n+1)}(\overset{(j)}{dM},\overset{(k)}{dJ}) &= K^{(n+1)}(\overset{(j)}{F},\overset{(k)}{S^*})
\end{aligned}
$$

Now write $dK^{(n)} = F^{(n)}dA + G^{(n)}dA^\dagger + H^{(n)}\,dt$. It follows from the bounds (108) that

$$\left\|F_t^{(n)}\right\| \leq \frac{\|J_t\|^{n-1}}{(n-1)!}\|S_t + F_t\|, \quad \left\|G_t^{(n)}\right\| \leq \frac{\|J_t\|^{n-1}}{(n-1)!}\|S_t^* + F_t^*\| \qquad (121)$$

$$\left\|H_t^{(n)}\right\| \leq \frac{\|J_t\|^{n-1}}{(n-1)!}\|U_t\| + \frac{1}{2}\frac{\|J_t\|^{n-2}}{(n-2)!}\|S_t\|^2 + \frac{\|J_t\|^{n-1}}{(n-1)!}\|F_t\|\cdot\|S_t\| + \frac{1}{2}\frac{\|J_t\|^{n}}{n!}\|F_t\|^2 \qquad (122)$$

It follows that $F^{(n)}, G^{(n)} \in L^2([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}})$ and $H^{(n)} \in L^1([0,1], \mathcal{B}(\mathfrak{H})_{\text{so}})$.

We now expand the terms on the right hand side of (116).

$$\begin{aligned}
i\int_0^1 & e^{i(1-u)(M+J)} d(M+J) e^{iu(M+J)}\,du \\
&= i\int_0^1 \left(\sum_{\alpha=0}^\infty i^\alpha (1-u)^\alpha K^{(\alpha)}((1-u))\right) d(M+J) \left(\sum_{\beta=0}^\infty i^\beta u^\beta K^{(\beta)}\right) du \\
&= \int_0^1 \sum_{n=0}^\infty i^{n+1} \sum_{\alpha+\beta=n} (1-u)^\alpha u^\beta K^{(\alpha)}((1-u)) d(M+J) K^{(\beta)}\,du \\
&= \sum_{n=0}^\infty \sum_{\alpha=0}^n i^{n+1} K^{(n+1)}(d\overset{(\alpha+1)}{(M+J)}) \\
&= \sum_{n=1}^\infty \sum_{\alpha=1}^n i^n K^{(n)}(\overset{(\alpha+1)}{dJ}) + \sum_{n=0}^\infty \sum_{\alpha=0}^n i^{n+1} K^{(n+1)}(\overset{(\alpha+1)}{dM}) \qquad (123)
\end{aligned}$$

Similarly

$$i^2 \int_{\Delta^{(2)}} e^{i(1-u_1)(M+J)} d(M+J) e^{i(u_1-u_2)(M+J)} d(M+J) e^{iu_2(M+J)}\,du \qquad (124)$$

$$\begin{aligned}
&= \sum_{n=0}^\infty \sum_{\alpha+\beta=0}^\infty i^{n+2} K^{(n+2)}(d\overset{(\alpha+1)}{(M+J)}, d\overset{(\alpha+\beta+2)}{(M+J)}) \\
&= \sum_{n=2}^\infty \sum_{1\leq j<k\leq n} i^n K^{(n)}(\overset{(j)}{dJ}, \overset{(k)}{dJ}) + \sum_{n=1}^\infty \sum_{1\leq j<k\leq n+1} i^{n+1} K^{(n+1)}(\overset{(j)}{dJ}, \overset{(k)}{dM}) \qquad (125) \\
&= \sum_{n=1}^\infty \sum_{1\leq j<k\leq n+1} i^{n+1} K^{(n+1)}(\overset{(j)}{dM}, \overset{(k)}{dJ}) + \sum_{n=0}^\infty \sum_{1\leq j<k\leq n+2} i^{n+2} K^{(n+2)}(\overset{(j)}{dM}, \overset{(k)}{dM})
\end{aligned}$$

Adding (123) and (125) and comparing with the expression (118) for $dK^{(n)}$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} i^n dK^{(n)} &= i\int_0^1 e^{i(1-u)(M+J)}d(M+J)e^{iu(M+J)}\,du \\ &+i^2\int_{\Delta^{(2)}} e^{i(1-u_1)(M+J)}d(M+J)e^{i(u_1-u_2)(M+J)}d(M+J)e^{iu_2(M+J)}\,du\end{aligned}$$

It follows from the bounds (121) and (122) that if

$$\kappa(t) = \|J_t\|\,e^{\|J_t\|}, \quad \beta(t) = 2e^{\|J_t\|}\,\|S_t + F_t\|\,,$$
$$\gamma(t) = e^{\|J_t\|}(\|U\|_t + \frac{1}{2}(\|S_t\| + \|F_t\|)^2$$

then the series $K^{(n)}$ and the quantum semimartingale N defined by

$$\begin{aligned}N_t &= I + \int_0^t i\int_0^1 e^{i(1-u)(M+J)}d(M+J)e^{iu(M+J)}\,du \\ +i^2 &\int_0^t\int_{\Delta^{(2)}} e^{i(1-u_1)(M+J)}d(M+J)e^{i(u_1-u_2)(M+J)}d(M+J)e^{iu_2(M+J)}\,du\end{aligned}$$

satisfy the conditions of Proposition 15.1. The quantum Duhamel formula (116) now follows from Proposition 15.1 (b) and (115).

Step 2. The integrands E, H, S and U are zero. In this case a product of differentials from $\{dM, dJ\}$ containing more than two dMs is zero as is any product containing the string $dJ \cdot dM \cdot dJ$.

Expanding $dK^{(n)}$ as in Step 1 and using the identity $d\Lambda \cdot d\Lambda = d\Lambda$ the sums (117) are replaced by

$$\sum_{k=1}^{n}\sum_{1\le j_1<\cdots<j_k\le n} K^{(n)}(\overset{(j_1)}{dJ},\ldots,\overset{(j_k)}{dJ}) = \sum_{k=1}^{n}\sum_{1\le j_1<\cdots<j_k\le n} K^{(n)}(\overset{(j_1)}{R},\ldots,\overset{(j_k)}{R})\,d\Lambda = E^{(n)\prime}\,d\Lambda.$$

It follows from the bounds (108) that

$$\left\|E^{(n)}{}'_t\right\| \le \sum_{r=1}^{n}\frac{\|J_t\|^{n-r}\,\|R_t\|^r}{(n-r)!r!}.$$

It follows from (113) that

$$\sum_0^\infty E^{(n)\prime} = e^{i(M+J+R)} - e^{i(M+J)}$$

the coefficient of $d\Lambda$ in the quantum Duhamel formula.

Using the identity $d\Lambda \cdot dA^\dagger = dA^\dagger$ the sum (118) is replaced by

$$\begin{aligned} & i\sum_{k=0}^{n} \sum_{1\le j_1<\cdots<j_{k+1}\le n+1} K^{(n+1)}(\overset{(j_1)}{dJ},\ldots,\overset{(j_k)}{dJ},\overset{(j_{k+1})}{dM}) \\ = \;& i\sum_{k=0}^{n} \sum_{1\le j_1<\cdots<j_{k+1}\le n+1} K^{(n+1)}(\overset{(j_1)}{R},\ldots,\overset{(j_k)}{R},\overset{(j_{k+1})}{F^*}) = iG^{(n)\prime}dA^\dagger. \end{aligned}$$

Using the identity $dA \cdot d\Lambda = dA$ the sum (119) is replaced by

$$\begin{aligned} & i\sum_{k=0}^{n} \sum_{1\le j_1<\cdots<j_{k+1}\le n+1} K^{(n+1)}(\overset{(j_1)}{dM},\ldots,\overset{(j_k)}{dJ},\overset{(j_{k+1})}{dJ}) \\ = \;& i\sum_{k=0}^{n} \sum_{1\le j_1<\cdots<j_{k+1}\le n+1} K^{(n+1)}(\overset{(j_1)}{F},\ldots,\overset{(j_k)}{R},\overset{(j_{k+1})}{R}) = iF^{(n)\prime}dA. \end{aligned}$$

Using the identity $dA \cdot d\Lambda \cdots d\Lambda dA^\dagger = dt$ the sum (120) is replaced by

$$\begin{aligned} & i^2\sum_{k=1}^{n} K^{(n+2)}(\overset{(j_1)}{dM},\overset{(j_2)}{dJ},\ldots,\overset{(j_k)}{dJ},\overset{(j_{k+1})}{dM}) \\ = \;& i^2\sum_{k=1}^{n} K^{(n+2)}(\overset{(j_1)}{F},\overset{(j_2)}{R},\ldots,\overset{(j_k)}{R},\overset{(j_{k+1})}{F^*}) = i^2H^{(n)\prime}dt \end{aligned}$$

It follows from the bounds (108) that

$$\left\|F_t^{(n)\prime}\right\|, \left\|G_t^{(n)\prime}\right\| \le \sum_{r=1}^{n} \frac{\|J_t\|^{n-r}\,\|R_t\|^r}{(n-r)!r!}\,\|F_t\|, \quad \left\|H_t^{(n)\prime}\right\| \le \frac{1}{2}\sum_{r=1}^{n} \frac{\|J_t\|^{n-r}\,\|R_t\|^r}{(n-r)!r!}\,\|F_t\|^2.$$

The coefficient of $dA^\dagger$ in the quantum Duhamel formula is

$$\begin{aligned} & i\int_0^1 e^{i(M+J+R)}F^* e^{i(M+J)}\,du \\ & = i\int_0^1 \sum_{\alpha=0}^{\infty}\sum_{k=0}^{\alpha} \sum_{1\le j_1<\cdots<j_k\le\alpha} i^\alpha(1-u)^\alpha K^{(\alpha)}((1-u);\overset{(j_k)}{R},\overset{(j_{k+1})}{R})F^* i^\beta u^\beta K^{(u)}\,du \\ & = \sum_{k=0}^{n} \sum_{1\le j_1<\cdots<j_{k+1}\le n+1} i^{n+1}K^{(n+1)}(\overset{(j_1)}{R},\ldots,\overset{(j_k)}{R},\overset{(j_{k+1})}{F^*}) \end{aligned}$$

Similarly the coefficient of dA is

$$i\int_0^1 e^{i(M+J)}Fe^{i(M+J+R)}\,du \;=\; i\sum_{k=0}^{n}\sum_{1\le j_1<\cdots<j_{k+1}\le n+1} i^{n+1}K^{(n+1)}(\overset{(j_1)}{F},\ldots,\overset{(j_k)}{R},\overset{(j_{k+1})}{R}).$$

The coefficient of dt is

$$\begin{aligned}
&i^2\int_{\delta^{(2)}}\sum_{\alpha=0}^{\infty}K^{(\alpha)}F\sum_{\beta=0}^{\infty}\sum_{k=0}^{\beta}\sum_{1\le j_1<\cdots<j_k\le\beta}K^{(\beta)}(u_1-u_2;\overset{(j_1)}{R},\ldots,\overset{(j_k)}{R})F^*\sum_{\gamma=0}^{\infty}K^{(\gamma)}(u_2)\,du\\
&=\;\sum_{n=0}^{\infty}i^{n+2}\sum_{\alpha+\beta+\gamma=n}\;\sum_{\alpha+1\le j_1<\cdots<j_k\le\alpha+\beta+1}K^{(\alpha+\beta+\gamma+2)}(\overset{(\alpha)}{F},\overset{(j_1)}{R},\ldots,\overset{(j_k)}{R},\overset{(\alpha+\beta+2)}{F^*}),\\
&=\;\sum_{n=0}^{\infty}\sum_{k=1}^{n}i^{n+2}K^{(n+2)}(\overset{(j_1)}{F},\overset{(j_2)}{R},\ldots,\overset{(j_k)}{R},\overset{(j_{k+1})}{F^*}).
\end{aligned}$$

The quantum Duhamel formula now follows using the same argument as in Step 2.

This method may also be used when E is non-zero but the calculations are very tedious and we leave them to the conscientious reader. ∎

16 Further Problems

We add five more problems to those posed in [28, §12].

This paper makes some progress with the problems in [28, §12] but provides nothing like a complete solution. The most important question is the following.

Problem 1. If M is an essentially self-adjoint quantum semimartingale does M satisfy the quantum Duhamel formula.

It is clear that both sides of (4) exist. Are they equal ?

Theorem 11.1 is not a generalisation of [28, Theorem 6.2] since not all regular quantum semimartingales satisfy its conditions. The conditions in Theorem 11.1 are on (E,F,G,H) while those in [28, Theorem 6.2] are on (E,F,G,H) *and* M. Moreover, a self-adjoint operator $T\in\mathcal{B}(\mathfrak{H})$ need not have $\ell_{00}\in\mathcal{A}(\mathcal{V}(T))$.

It is possible to construct a synthetic theorem containing Theorem 11.1 and Theorem 6.2 of [28], but this would be unsatisfactory.

Problem 3. Find natural conditions on a quantum semimartingale M for which the conclusions of both Theorem 11.1 *and* [28, Theorem 6.2.] *remain true.*

In general the cmx representation of the quantum stochastic process F considered in Section 13 is complicated and does not satisfy the conditions of Corollary 11.3. In that case the classical Ito formula (105) does not follow from (102). If M is irregular the classical Ito formula for M does not follow from [28, Theorem 6.2].

Problem 3. Find conditions on an integrable quadruple $(0, F, F^*, 0)$, which is not necessarily commutative, which ensure that the quantum Ito formula (102) *is valid and implies the classical Ito formula* (105).

Formally, the analysis in Section 13 carries over to the case when M is an essentially self-adjoint quantum semimartingale on $\mathcal{E}$ which satisfies the quantum Ito formula. In this case h' and h'' must be replaced by Dh and $D_I^2 h$. There are serious technical difficulties in finding bounds on Dh and $D_I^2 h$ which we have been unable to overcome. See for example [22].

Problem 4. Is it possible to modify the method used in Section 13 to give a more analytic proof of Theorem 15.3 ?

The final problem is about the Stratonovich formula.

Problem 5. Is the cmx Stratonovich formula (104) valid for $f \in C^{3+}(\mathbb{R})$ and does it give rise to the corresponding quantum Stratonovich formula (103) ?

References

[1] S. ATTAL, 'An algebra of non-commutative bounded semimartingales: square and angle quantum brackets', Journal of Functional Analysis, 124 (1994) 292–332.

[2] S. ATTAL, 'Classical and quantum stochastic calculus', *Prepublication de l'Institute Fourier, Grenoble, 358 (1998)*

[3] S. ATTAL and P. A. MEYER, 'Interprétation probabiliste et extension des intégrales stochastiques non commutatives', Springer Lecture Notes in Mathematics, 1557, (1993) 312–327.

[4] O. BRATTELI and D. W. ROBINSON, *Operator algebras and quantum statistical mechanics I* (Springer–Verlag, 1979).

[5] O. BRATTELI and D. W. ROBINSON, *Operator algebras and quantum statistical mechanics II* (Springer–Verlag, 1979).

[6] J. DIESTEL and J. J. UHL, *Vector measures* (American Mathematical Society, 1977).

[7] M. P. EVANS, 'Existence of quantum diffusions', Probab. Theory Related Fields 81 (1989) 473–483.

[8] C. W. GARDINER and M. J. COLLETT, Input and output in damped quantum systems: 'Quantum stochastic differential equations and the master equation', Phys. Rev. A 31 (6) (1985) 3761–3773.

[9] J. GLIMM and A JAFFE, *Quantum Physics, A Functional Integral Point of View,* (Springer-Verlag, New York 1981).

[10] E. HILLE and R. S. PHILLIPS, *Functional Analysis and Semigroups,* (Colloq. Publ. Amer. Math. Soc. 1957).

[11] R. L. HUDSON and K. R. PARTHASARATHY, 'Quantum Ito's formula and stochastic evolutions', Comm. Math. Phys. 93, (1984) 301–323.

[12] I. KARATZAS and S. E. SHREVE, *Brownian motion and stochastic calculus* (Springer-Verlag 1988).

[13] H. KUNITA and S. WATANABE, 'On square integrable martingales', Nagoya Math. J. 30 (1967) 209–245.

[14] J. M. LINDSAY, 'Quantum and non-causal stochastic calculus', Probab. Theory Related Fields 97 (1993) 65–80.

[15] J. M. LINDSAY, 'Independence for quantum stochastic integrators', *Quantum Probability and Related Topics, Vol. VI,* (ed. L. Accardi) (World Scientific, Singapore, 1991) 325–332.

[16] P. A. MEYER, *Quantum probability for probabilists* (Lecture Notes in Mathematics 1538, Springer-Verlag 1993).

[17] E. NELSON, 'Analytic vectors', Ann. Math. 70 (1959) 572–615.

[18] N. OBATA, *White noise calculus and Fock space* (Lecture Notes in Mathematics 1577, Springer-Verlag 1993).

[19] N. OBATA, 'Generalised quantum stochastic processes on Fock space, Publ. RIMS, Kyoto Univ. 31 (1995) 667–702.

[20] K. R. PARTHASARATHY, *An introduction to quantum stochastic calculus* (Birkhauser, 1992).

[21] K. R. PARTHASARATHY, 'Quantum stochastic calculus', in *Proceedings of Stochastic processes and their applications, Nagoya 1985,* (ed. K Ito and T Hida) Springer Lecture Notes in Mathematics 1203, (1986) 177–196.

[22] V. V. Peller, Lecture Notes in Pure and Applied Mathematics 122, (1990).

[23] M. REED and B. SIMON, *Methods of Modern Mathematical Physics, I and II* (Academic Press, 1972 and 1975).

[24] L. C. G. ROGERS and D. WILLIAMS, *Diffusions, Markov processes, and martingales: Ito calculus* (John Wiley and Sons, 1987).

[25] W. RUDIN, *Real and Complex Analysis* (McGraw-Hill 1970).

[26] T. SAITO and T. ARIMITSU, 'Quantum stochastic Liouville equation of Ito type', Modern Phys. Lett. B, 7 (29-30) (1993) 1951–1959.

[27] G. F. VINCENT–SMITH, 'Unitary quantum stochastic evolutions', Proc. London Math. Soc. (3) 63 (1991) 401–425.

[28] G. F. VINCENT–SMITH, 'The Ito formula for quantum semimartingales', Proc. London Math. Soc. (3) 75 (1997) 671–720.

[29] G. F. VINCENT–SMITH, 'The Ito formula for perturbed Brownian martingales', Bull. London Math. Soc. 32 (2000) 736–745.

[30] J. WEIDMANN, *Linear Operators in Hilbert Spaces* (Springer-Verlag, 1980).

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